Advances in Mathematics **136**, 141–182 (1998)
Article No. AI981729

On Quiver Varieties

G. Lusztig*

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Received November 13, 1997; accepted January 3, 1998

Contents.

INTRODUCTION

0.1. Let Γ be a finite subgroup $\neq \{1\}$ of $SL(T)$ where T is a two-dimensional $\mathbf{C}$-vector space. Let U be the enveloping algebra of the affine Lie algebra with symmetric Cartan datum corresponding to Γ under McKay's correspondence [M] and let U^- be the lower triangular part of U.

In [L1, L2] I gave a construction of U^- together with a basis of it, directly in terms of Γ. (The assumption of [L2] that Γ has even order is not necessary for what follows.) Let S^u be the u-th symmetric power of T and let $S^\dagger = \bigoplus_u S^u$ be the symmetric algebra of T. Let M' be a finite dimensional Γ-module over $\mathbf{C}$. Let $\Lambda_{M'}$ be the set of all $S^\dagger$-algebra structures on M' that are compatible with the natural Γ-action and are such that S^u acts as zero for large enough u. In [L1, L2] it was shown that $\Lambda_{M'}$ is in a natural way an affine algebraic variety of pure dimension. Moreover, for each irreducible component X of $\Lambda_{M'}$, a canonical constructible function $f_X: \Lambda_{M'} \to \mathbf{Z}$ was defined. It was shown that the $\mathbf{C}$-vector space spanned by the f_X (for various M' up to isomorphism and various X) is closed under a natural convolution operation (similar to one considered by Hall and Ringel) and is in fact an algebra isomorphic to U^- for which the f_X provide a

* Supported in part by the National Science Foundation.

0001-8708/98 $25.00

distinguished basis B (not necessarily the same as the canonical basis). The f_X corresponding to a fixed M' span a weight space of U^-.

0.2. Nakajima [N1, N2] has found a beautiful modification of the variety $\Lambda_{M'}$. Let M be a second finite dimensional Γ-module over $\mathbf{C}$. Nakajima defines an open set ("stable points") $\Lambda^{sn}_{M, M'}$ of $\Lambda_{M'} \times \mathrm{Hom}_\Gamma(M, M')$ on which the automorphism group G of the Γ-module M' acts freely. He shows that the quotient $\Lambda^{sn}_{M, M'}/G$ is a projective algebraic variety and that it plays the same role for the study of highest weight integrable modules of U as $\Lambda_{M'}$ played for the study of U^-.

Thus, let L be an integrable highest weight U-module. We may assume that for any irreducible representation ρ of Γ, the highest weight evaluated at the simple coroot corresponding to ρ is the multiplicity of 1 in the Γ-module $\rho \otimes M$. It is easy to see that, by applying the elements of B to the highest weight vector of L, we get a bijection of a subset B_M of B onto a basis of L. (The elements in $B - B_M$ are mapped to 0.) Nakajima shows that an irreducible component X of $\Lambda_{M'}$ corresponds to an element of B_M if and only if $X \times \mathrm{Hom}_\Gamma(M, M')$ meets $\Lambda^{sn}_{M, M'}$. Hence B_M is in natural bijection with the set of irreducible components of the variety $\Lambda^{sn}_{M, M'}/G$ (with variable M').

0.3. From the fact that $\Lambda_{M'}$ has pure dimension, it follows that $\Lambda^{sn}_{M, M'}/G$ has pure dimension. More precisely, Nakajima shows that $\Lambda^{sn}_{M, M'}/G$ is a lagrangian subvariety of a symplectic manifold $\Lambda^{s}_{M, M', 0}/G$ (again an orbit space). He also shows that $\Lambda^{sn}_{M, M'}/G$ is just one fibre of a canonical proper morphism $\Lambda^{s}_{M, M', 0}/G \to Y_{M, M'}$ where $Y_{M, M'}$ is a certain variety defined as a geometric quotient.

0.4. Until now there were no explicit descriptions of the "quiver variety" $\Lambda^{sn}_{M, M'}/G$ other than as an orbit space (except in some very special cases). In this paper, we give the following explicit description of the quiver variety $\Lambda^{sn}_{M, M'}/G$. We show (see Section 6) that $\Lambda^{sn}_{M, M'}/G$ is naturally in bijection with the variety $\mathscr{H}^{M'}_0(M)$ consisting of all $S^\dagger$-submodules $\mathscr{W}$ of $S^\dagger \otimes M$ which are also Γ-submodules such that $(S^\dagger \otimes M)/\mathscr{W}$ is isomorphic to M' as a Γ-module and such that $\mathscr{W}$ contains $S^u \otimes M$ for all large enough u. (Γ acts on both factors of $S^\dagger \otimes M$.) In this description, the fact that $\Lambda^{sn}_{M, M'}/G$ is a projective variety is obvious.

0.5. The results above apply also when U, U^- are replaced by the analogous objects U_*, U^-_* attached to the corresponding finite dimensional simple Lie algebra (of simply laced type). See Section 7. We simply have to restrict ourselves to Γ-modules M, M' in which the unit representation of Γ does not occur. Thus, the varieties $\Lambda_{M'}$ (resp. $\Lambda^{sn}_{M, M'}/G = \mathscr{H}^{M'}_0(M)$)

for such M, M' describe U_*^- (resp. the finite dimensional highest weight modules for U_*). These are just a subset of the set of varieties considered above in the affine case.

0.6. For fixed M as in 0.5, the varieties $Y_{M,M'}$ in 0.3 (for various M' as in 0.5) form naturally an inductive system, so their union Y_M is well defined. We will identify Y_M with an explicitly defined algebraic variety $Z^0_{\mathbf{D}}$, at least up to homeomorphism. (See Section 5.) The proof in Section 5 is based on a result in invariant theory from Section 1.

The analogue of Y_M in the affine case is not described in this paper.

0.7. Most of this paper is written so that it applies to a general graph, not just one of affine or finite type. In the general case a central role is played by the *pre-projective algebra* introduced by Gelfand and Ponomarev [GP] and further studied in [DR]. This algebra is spanned by the paths in the graph (we are allowed to walk along an edge in either direction) and there are certain quadratic relations, one for each vertex in the graph. An explicit description of the quiver varieties in this case is given in 2.26. Another explicit description is given in 4.13 for graphs satisfying an evenness condition. (The proof is based on techniques from Section 3.)

1. A RESULT ON INVARIANT POLYNOMIALS

1.1. We fix an oriented graph (I, H) where I, H are finite sets. Thus, we are given two maps $H \to I$ denoted $h \mapsto h'$ and $h \mapsto h''$. We say that I is the set of vertices and H is the set of oriented edges (the edge $h \in H$ goes from the vertex h' to the vertex h'').

1.2. Let $\mathscr{C}$ be the category whose objects are I-graded $\mathbf{C}$-vector spaces $\mathbf{V} = \bigoplus_{i \in I} \mathbf{V}_i$ and whose morphism are linear maps compatible with the grading. Let $\mathscr{C}^0$ be the full subcategory of $\mathscr{C}$ whose objects are I-graded finite dimensional $\mathbf{C}$-vector spaces.

Given $\mathbf{D} \in \mathscr{C}^0, \mathbf{V} \in \mathscr{C}$ we define $\mathbf{M} = \mathbf{M}_{\mathbf{D},\mathbf{V}}$ to be the $\mathbf{C}$-vector space consisting of all triples (x, p, q) where

$$x = (x_h)_{h \in H} \in \bigoplus_{h \in H} \mathrm{Hom}(\mathbf{V}_{h'}, \mathbf{V}_{h''}),$$

$$p = (p_i)_{i \in I} \in \bigoplus_{i \in I} \mathrm{Hom}(\mathbf{D}_i, \mathbf{V}_i),$$

$$q = (q_i)_{i \in I} \in \bigoplus_{i \in I} \mathrm{Hom}(\mathbf{V}_i, \mathbf{D}_i).$$

In the remainder of this section we assume that $\mathbf{V} \in \mathscr{C}^0$. There is a natural linear action of the algebraic group

$$G = G_{\mathbf{V}} = \prod_{i \in I} GL(\mathbf{V}_i)$$

on $\mathbf{M}$ given by

$$g\colon (x, p, q) \mapsto (gx, gp, qg^{-1}),$$

where $(gx)_h = g_{h''} x_h g_{h'}^{-1}$ for all h, $(gp)_i = g_i p_i$, $(qg^{-1})_i = q_i g_i^{-1}$ for all i.

Let $\mathscr{R}$ be the algebra of regular functions $\mathbf{M} \to \mathbf{C}$. The action of G on $\mathbf{M}$ induces an action of G on $\mathscr{R}$. Let $\mathscr{R}^G$ be the algebra of invariants of G on $\mathscr{R}$. We want to describe a set of generators of the algebra $\mathscr{R}^G$. We give two examples of elements in $\mathscr{R}^G$.

(a) Let $h_1, h_2, ..., h_r$ be a *cycle* of our graph, that is a sequence in H such that $h_1'' = h_2'$, $h_2'' = h_3'$, ..., $h_{r-1}'' = h_r'$, $h_r'' = h_1'$. This cycle defines a G-invariant polynomial $\mathbf{M} \to \mathbf{C}$ given by $(x, p, q) \mapsto \mathrm{Tr}(x_{h_r} x_{h_{r-1}} \cdots x_{h_1}\colon \mathbf{V}_{h_1'} \to \mathbf{V}_{h_1'})$.

(b) Let $h_1, h_2, ..., h_r$ be a *path* of our graph, that is a sequence in H such that $h_1'' = h_2'$, $h_2'' = h_3'$, ..., $h_{r-1}'' = h_r'$. This path together with a linear form χ on $\mathrm{Hom}(\mathbf{D}_{h_1'}, \mathbf{D}_{h_r''})$ defines a G-invariant polynomial $\mathbf{M} \to \mathbf{C}$ given by $(x, p, q) \mapsto \chi(q_{h_r''} x_{h_r} x_{h_{r-1}} \cdots x_{h_1} p_{h_1'})$.

THEOREM 1.3. *The G-invariant functions described in* 1.2(a),(b) *form a set of algebra generators of* $\mathscr{R}^G$.

For the proof of the theorem we need a number of lemmas.

LEMMA 1.4. *Let V' be a finite dimensional* $\mathbf{C}$*-vector space, let Γ be a group and let $\Gamma \to GL(V')$ be a homomorphism. Let n be an integer $\geqslant 1$. Consider the map from the space of linear maps $V'^{\otimes n} \to \mathbf{C}$ to the space of homogeneous polynomials $V' \to \mathbf{C}$ of degree n given by $f \mapsto f_1$ where $f_1(v') = f(v' \otimes v' \otimes \cdots \otimes v')$. This map is surjective. Moreover, its restriction to the spaces of Γ-invariants of the two spaces above is also surjective.*

If $f_1\colon V' \to \mathbf{C}$ is a homogeneous polynomial of degree n, we define $f_2\colon V'^n \to \mathbf{C}$ by

$$f_2(v_1', v_2', ..., v_n') = (n!)^{-1} \sum (-1)^{n-r} f_1(v_{j_1}' + v_{j_2}' + \cdots + v_{j_r}'),$$

where the sum is taken over all sequences $1 \leqslant j_1 < j_2 < \cdots < j_r \leqslant n$ with $r > 0$. One can check that f_2 is a multilinear function hence it defines a linear form

$V'^{\otimes n} \to \mathbf{C}$ denoted again by f_2. The map of the lemma attaches to f_2 the polynomial f_1. This follows from the identity

$$\sum_{r=1}^{n} \binom{n}{r} (-1)^{n-r} r^n = n!. \tag{a}$$

Assuming that (a) is proved, we see that the first statement of the lemma holds. The second one follows from the first since the construction $f_1 \mapsto f_2$ commutes with the action of Γ.

We now prove (a). Let J be a set of cardinal n. For any $S \subset J$, let X_S be the set of all maps $J \to J$ with image contained in S and let Y_S be the set of all maps $J \to J$ with image equal to S. We have $X_S = \bigsqcup_{T;\, T \subset S} Y_S$ hence $\sharp(X_S) = \sum_{T;\, T \subset S} \sharp(Y_T)$. From this we deduce that $\sharp(Y_S) = \sum_{T;\, T \subset S} (-1)^{\sharp S - \sharp T} \sharp(X_T)$. We take $S = J$ in the last equality. Note that $\sharp(Y_J) = n!$ and $\sharp(X_T) = \sharp(T)^n$. This proves (a). The lemma is proved.

LEMMA 1.5. *For each $i \in I$ let E^i be a finite dimensional* **C**-*vector space, let G_i be a reductive group over* **C** *and let $G_i \to GL(E^i)$ be a homomorphism of algebraic groups. We regard $E = \bigotimes_{i \in I} E^i$ naturally as a $\prod_{i \in I} G_i$-module. Let E_0^i (resp. E_0) be the space of linear forms on E^i (resp. E) that are invariant under G_i (resp. $\prod_{i \in I} G_i$). Then we have canonically $\bigotimes_{i \in I} E_0^i \cong E_0$.*

The proof is immediate.

LEMMA 1.6. *Let W be a finite dimensional* **C**-*vector space. Let W^* be the dual vector space and let $\langle\ ,\ \rangle\colon W \times W^* \to \mathbf{C}$ be the canonical pairing. We consider the tensor product $T = W^{\otimes n} \otimes W^{*\otimes m}$.*

(a) *If $m \neq n$, then the space of $GL(W)$-invariant linear forms on T is* 0.

(b) *Assume that $m = n$. For any permutation $\sigma\colon [1, n] \to [1, n]$ let $f_\sigma\colon T \to \mathbf{C}$ be the $GL(W)$-invariant linear form on T given by*

$$f_\sigma(w_1 \otimes w_2 \otimes \cdots w_n \otimes w'_1 \otimes w'_2 \otimes \cdots w'_n) = \prod_{s=1}^{n} \langle w_s, w'_{\sigma(s)} \rangle.$$

Then the f_σ for the various permutations σ as above generate the vector space of $GL(W)$-invariant linear forms on T.

This is well known. See [W].

1.7. *Proof of Theorem* 1.3. We fix an integer $n \geqslant 1$. It is enough to show that the space of G-invariant homogeneous polynomials $\mathbf{M} \to \mathbf{C}$ of degree n is generated as a vector space by the various products of functions of the form 1.2(a),(b) which have degree n.

For each $i \in I$, let $\mathscr{D}_i$ be a basis of the vector space $\mathbf{D}_i$. We can identify

$$\mathbf{M} = \Big(\bigoplus_{h \in H} \mathbf{V}^*_{h'} \otimes \mathbf{V}_{h''}\Big) \oplus \Big(\bigoplus_{i \in I;\, \delta \in \mathscr{D}_i} \otimes \mathbf{V}_{i,\delta}\Big) \oplus \Big(\bigoplus_{i \in I;\, \delta \in \mathscr{D}_i} \mathbf{V}^*_{i,\delta}\Big)$$

as G-modules, where G acts on the right hand side in the obvious way. Here $\mathbf{V}_{i,\delta}$ is a copy of $\mathbf{V}_i$ indexed by δ. Then $\mathbf{M}^{\otimes n}$ is a direct sum of G-stable subspaces of the form $E = E_1 \otimes E_2 \otimes \cdots \otimes E_n$ where each E_j is either $\mathbf{V}^*_{h'} \otimes \mathbf{V}_{h''}$ for some $h \in H$, or $\mathbf{V}_{i,\delta}$ for some $i \in I$, $\delta \in \mathscr{D}_i$, or $\mathbf{V}^*_{i,\delta}$ for some $i \in I$, $\delta \in \mathscr{D}_i$. Hence the space of G-invariant linear forms $\mathbf{M}^{\otimes n} \to \mathbf{C}$ is naturally the direct sum of the spaces of G-invariant linear forms $E \to \mathbf{C}$ for various E as above.

We fix E as above. We consider a second copy $I^\flat$ of I in bijection with i under $i \leftrightarrow i^\flat$. For each $j \in [1, n]$ we define a subset $\mathscr{S}_j \subset I \sqcup I^\flat$ as follows:

$\mathscr{S}_j = \{h'^\flat, h''^\flat\}$ if $E_j = \mathbf{V}^*_{h'} \otimes \mathbf{V}_{h''}$ for $h \in H$;
$\mathscr{S}_j = \{i\}$ if $E_j = \mathbf{V}_{i,\delta}$;
$\mathscr{S}_j = \{i^\flat\}$ if $E_j = \mathbf{V}^*_{i,\delta}$.

Let $\mathscr{S} = \bigcup_{j \in [1,n]} \mathscr{S}_j$. We regard $\mathscr{S}$ as a multisubset of $I \sqcup I^\flat$, that is i (resp. $i^\flat$) appears in $\mathscr{S}$ as many times as the number of j with $i \in \mathscr{S}_j$ (resp. $i^\flat \in \mathscr{S}_j$). An *arrangement* for $\mathscr{S}$ is a partition of $\mathscr{S}$ into two-element subsets of the form $\{i, i^\flat\}$. If an arrangement for $\mathscr{S}$ exists, we choose one and we associate to it a linear form $\gamma\colon E \to \mathbf{C}$ as follows. Let $e_j \in E_j$ be such that

$$e_j = e'_j \otimes e''_j, \qquad e'_j \in \mathbf{V}^*_{h'}, \quad e''_j \in \mathbf{V}_{h''} \quad \text{whenever} \quad E_j = \mathbf{V}^*_{h'} \otimes \mathbf{V}_{h''} \quad \text{for} \quad h \in H.$$

Then $\gamma(e_1 \otimes e_2 \otimes \cdots \otimes e_n)$ is by definition a product of factors, one for each two-element subset $\{i, i^\flat\}$ in the arrangement. Recall that $i \in \mathscr{S}_j$, $i^\flat \in \mathscr{S}_{\tilde{j}}$ for well defined $j, \tilde{j} \in [1, n]$. The corresponding factor is

$\langle e_j, e_{\tilde{j}}\rangle$ if $E_j = \mathbf{V}_{i,\delta}$, $\quad E_{\tilde{j}} = \mathbf{V}^*_{i,\tilde{\delta}}$;
$\langle e''_j, e_{\tilde{j}}\rangle$ if $E_j = \mathbf{V}^*_{h'} \otimes \mathbf{V}_{h''}$, $\quad h'' = i$, $\quad E_{\tilde{j}} = \mathbf{V}^*_{i,\tilde{\delta}}$;
$\langle e_j, e'_{\tilde{j}}\rangle$ if $E_j = \mathbf{V}_{i,\delta}$, $\quad E_{\tilde{j}} = \mathbf{V}^*_{\tilde{h}'} \otimes \mathbf{V}_{\tilde{h}''}$, $\quad \tilde{h}' = i$;
$\langle e''_j, e'_{\tilde{j}}\rangle$ if $E_j = \mathbf{V}^*_{h'} \otimes \mathbf{V}_{h''}$, $\quad h'' = i$, $\quad E_{\tilde{j}} = \mathbf{V}^*_{\tilde{h}'} \otimes \mathbf{V}_{\tilde{h}''}$, $\quad \tilde{h}' = i$.

It is clear that the linear form γ is well-defined and G-invariant.

We can consider the elements $j \in [1, n]$ as the vertices of a graph Δ in which $j, \tilde{j}$ are joined by an edge if there exist $i \in \mathscr{S}_j$, $i^\flat \in \mathscr{S}_{\tilde{j}}$ so that $\{i, i^\flat\}$ belongs to our arrangement. We decompose $[1, n]$ in a disjoint union of subsets: the connected components of Δ. Each two-element subset in our arrangement is associated as above to an edge of Δ, hence it is attached to some connected component.

Grouping together the factors in $\gamma(e_1 \otimes e_2 \otimes \cdots \otimes e_n)$ corresponding to $\{i, i^\flat\}$ that are attached to a fixed connected component, we obtain a decomposition of $\gamma(e_1 \otimes e_2 \otimes \cdots \otimes e_n)$ as a product of factors, one for each connected component.

We now describe the factor corresponding to a fixed connected component Δ_0. There are two cases:

(a) Δ_0 contains no vertices j such that $E_j = \mathbf{V}_{i,\delta}$ or $E_j = \mathbf{V}^*_{i,\delta}$;

(b) Δ_0 contains exactly one vertex j such that $E_j = \mathbf{V}_{i,\delta}$ and exactly one vertex j' such that $E_{j'} = \mathbf{V}^*_{i',\delta'}$.

In case (a), there exists a cycle $h_1, h_2, ..., h_r$ in (I, H) such that the Δ_0 is a subset $\{j_1, j_2, ... j_r\}$ of $[1, n]$ where $E_{j_s} = \mathbf{V}^*_{h'_s} \otimes \mathbf{V}_{h''_s}$ for $s = 1, ..., r$ and the corresponding factor is

$$\langle e''_{j_1}, e'_{j_2} \rangle \langle e''_{j_2}, e'_{j_3} \rangle \cdots \langle e''_{j_{r-1}}, e'_{j_r} \rangle \langle e''_{j_r}, e'_{j_1} \rangle.$$

In case (b), there exists a path $h_1, h_2, ..., h_r$ in (I, H) and $i, i' \in I$, $\delta \in \mathscr{D}_i$, $\delta' \in \mathscr{D}_{i'}$ such that Δ_0 is a subset $\{j, j_1, j_2, ..., j_r, j'\}$ of $[1, n]$ where

$$E_{j_s} = \mathbf{V}^*_{h'_s} \otimes \mathbf{V}_{h''_s} \qquad \text{for} \quad s = 1, ..., r, \qquad E_j = \mathbf{V}_{i,\delta}, E_{j'} = \mathbf{V}^*_{i',\delta'}$$

and the corresponding factor is

$$\langle e_j, e'_{j_1} \rangle \langle e''_{j_1}, e'_{j_2} \rangle \langle e''_{j_2}, e'_{j_3} \rangle \cdots \langle e''_{j_{r-1}}, e'_{j_r} \rangle \langle e''_{j_r}, e_{j'} \rangle.$$

Note that $E = \bigotimes_{i \in I} E^i$ where each E^i is a tensor product of factors of form V_i and V_i^*. Applying Lemma 1.5 to $E = \bigotimes_{i \in I} E^i$ and $G_i = GL(\mathbf{V}_i)$ and Lemma 1.6 to $W = \mathbf{V}_i$ and $T = E^i$ we see that the space of G-invariant linear forms $E \to \mathbf{C}$ is spanned by the linear forms corresponding to arrangements as above. (If no arrangements exist, the space of G-invariant linear forms $E \to \mathbf{C}$ is zero.) These linear forms may be regarded as linear forms $\mathbf{M}^{\otimes n} \to \mathbf{C}$ which are zero on summands other than E; and these generate the space of G-invariant linear forms $\mathbf{M}^{\otimes n} \to \mathbf{C}$. From these we obtain a set of generators for the space of G-invariant homogeneous polynomials $\mathbf{M} \to \mathbf{C}$ of degree n. These are products of factors corresponding to the components described in (a), (b); the factors are of the type given in 1.2(a), (b). The theorem is proved.

1.8. *Remark.* In the special case where $\mathbf{D} = 0$, the theorem reduces to the following result of Le Bruyn and Procesi [LP].

COROLLARY 1.9. *If* $\mathbf{D} = 0$, *the G-invariant functions described in* 1.2(a) *form a set of algebra generators of* $\mathscr{R}^G$.

2. THE MODULI SPACE OF STABLE QUADRUPLES

2.1. In the remainder of this paper we assume that the data in 1.1 satisfy $h' \neq h''$ for all $h \in H$ and that we are given a (necessarily fixed point free) involution $h \mapsto \bar{h}$ of H such that $(\bar{h})' = h''$ for all $h \in H$. We assume that a function ε: $H \to \mathbf{C}^*$ such that

$$\varepsilon(h) + \varepsilon(\bar{h}) = 0 \tag{a}$$

for all $h \in H$ is fixed. (See [L1, 12.15] for a discussion of the reason why all choices of ε are essentially equivalent.)

For $v \in \mathbf{Z}[I]$ we shall write v_i for the i-component of v. Thus, $v_i \in \mathbf{Z}$ and $v = \sum_{i \in I} v_i i$. For $\mathbf{V} \in \mathscr{C}$ and $v \in \mathbf{N}[I]$, the notation $|\mathbf{V}| = v$ means: $\mathbf{V} \in \mathscr{C}^0$ and $\dim \mathbf{V}_i = v_i$ for all i.

2.2. For any $i, j \in I$, let $\mathscr{P}^i_j$ be the set of all sequences $h_1, h_2, ..., h_r$ in H such that $i = h''_1, h'_1 = h''_2, h'_2 = h''_3, ..., h'_{r-1} = h''_r, h'_r = j$. (Then $h_r, h_{r-1}, ..., h_1$ is a path in the sense of 1.2.) Let $e_i \in \mathscr{P}^i_i$ be the empty sequence. For i, j, k in I, we have a well defined map

$$\mathscr{P}^i_j \times \mathscr{P}^j_k \to \mathscr{P}^i_k \tag{a}$$

given by

$$(h_1, h_2, ..., h_r), (\tilde{h}_1, \tilde{h}_2, ..., \tilde{h}_{r'}) \mapsto (h_1, h_2, ..., h_r, \tilde{h}_1, \tilde{h}_2, ..., \tilde{h}_{r'}).$$

Let $\mathscr{F}^i_j$ be the $\mathbf{C}$-vector space with basis $\mathscr{P}^i_j$. Let $\mathscr{F} = \bigoplus_{i,j} \mathscr{F}^i_j$. We regard $\mathscr{F}$ as a $\mathbf{C}$-algebra in which the product $f^i_j, f^{j'}_k$ of $f^i_j \in \mathscr{P}^i_j$, $f^{j'}_k \in \mathscr{P}^{j'}_k$ is the image of $(f^i_j, f^{j'}_k)$ under (a), if $j = j'$ and is 0, if $j' \neq j$. This algebra is associative and has unit element $\sum_{i \in I} e_i$. The elements of $\mathscr{F}$ of the form $(h_1, h_2, ..., h_r) \in \mathscr{P}^i_j$ are said to be *monomials* of length r. In particular, the e_i are monomials of length 0.

2.3. Let $\mathbf{D} \in \mathscr{C}^0$. Let $\mathscr{E}'^{\mathbf{D}} = \bigoplus_{i \in I} \mathscr{E}'^{\mathbf{D}}_i$, $\mathscr{E}^{\mathbf{D}} = \bigoplus_{i \in I} \mathscr{E}^{\mathbf{D}}_i$, $\mathscr{E}''^{\mathbf{D}} = \bigoplus_{i \in I} \mathscr{E}''^{\mathbf{D}}_i$ be the I-graded vector spaces defined by

$$\mathscr{E}'^{\mathbf{D}}_i = \bigoplus_{j,k \in I} \mathscr{F}^i_j \otimes \mathscr{F}^j_k \otimes \mathbf{D}_k,$$

$$\mathscr{E}^{\mathbf{D}}_i = \bigoplus_{k \in I} \mathscr{F}^i_k \otimes \mathbf{D}_k, \qquad \mathscr{E}''^{\mathbf{D}}_i = \bigoplus_{k \in I} \operatorname{Hom}(\mathscr{F}^k_i, \mathbf{D}_k).$$

We regard $\mathscr{E}'^{\mathbf{D}}$ as a $\mathscr{F}$-module by setting for $f^n_{i'} \in \mathscr{F}^n_{i'}$, $f^i_j \in \mathscr{F}^i_j$, $f^j_k \in \mathscr{F}^j_k$, $d_k \in \mathbf{D}_k$

$$f^n_{i'}(f^i_j \otimes f^j_k \otimes d_k) = (f^n_{i'} f^i_j) \otimes f^j_k \otimes d_k.$$

We regard $\mathscr{E}^{\mathbf{D}}$ as a $\mathscr{F}$-module by setting for $f^n_{i'} \in \mathscr{F}^n_{i'}$, $f^i_k \in \mathscr{F}^i_k$, $d_k \in \mathbf{D}_k$

$$f^n_{i'}(f^i_k \otimes d_k) = (f^n_{i'} f^i_k) \otimes d_k.$$

We regard $\mathscr{E}''^{\mathbf{D}}$ as a $\mathscr{F}$-module as follows. For $f^n_{i'} \in \mathscr{F}^n_{i'}$ and $\phi \in \mathrm{Hom}(\mathscr{F}^k_i, \mathbf{D}_k)$ we set $f^n_{i'}\phi = 0$ (if $i' \neq i$) and $f^n_{i'}\phi = \phi' \in \mathrm{Hom}(\mathscr{F}^k_n, \mathbf{D}_k)$ (if $i' = i$) where $\phi'(f^k_n) = \phi(f^k_n f^n_{i'})$ for $f^k_n \in \mathscr{F}^k_n$. Note that, for $E = \mathscr{E}'^{\mathbf{D}}, \mathscr{E}^{\mathbf{D}}, \mathscr{E}''^{\mathbf{D}}$ we have $e_i E_i = E_i$ and $e_j E_i = 0$ if $i \neq j$.

2.4. Let $\lambda = \sum_{i \in I} \lambda_i i \in \mathbf{C}[I]$, where $\lambda_i \in \mathbf{C}$ for all $i \in I$. For $i \in I$ we define

$$\theta_i = \sum_{h \in H;\, h'' = i} \varepsilon(h)(h, \bar{h}) \in \mathscr{F}^i_i, \qquad \theta_{i,\lambda} = \theta_i - \lambda_i e_i \in \mathscr{F}^i_i.$$

Here we regard $(h, \bar{h})$ as an element of $\mathscr{P}^i_i$.

A vector in the vector space

$$\prod_{i, j \in I} \mathrm{Hom}(\mathscr{F}^i_j, \mathrm{Hom}(\mathbf{D}_j, \mathbf{D}_i)) \tag{a}$$

may be considered as a collection

$$\pi = \{\pi_{f^i_j} \in \mathrm{Hom}(\mathbf{D}_j, \mathbf{D}_i) \mid f^i_j \in \mathscr{F}^i_j;\, i, j \in I\}$$

such that $\pi_{f^i_j}$ depends linearly of $f^i_j \in \mathscr{F}^i_j$.

Let $Z^\lambda_{\mathbf{D}}$ be the set of vectors π in the vector space (a) such that

$$\pi_{f^j_i} \pi_{f^i_k} = \pi_{f^j_i \theta_{i,\lambda} f^i_k} \tag{b}$$

for all $f^j_i \in \mathscr{F}^j_i$, $f^i_k \in \mathscr{F}^i_k$.

2.5. Let $\pi \in Z^\lambda_{\mathbf{D}}$. We define a $\mathbf{C}$-linear map $\alpha_\pi \colon \mathscr{E}^{\mathbf{D}} \to \mathscr{E}''^{\mathbf{D}}$ by

$$\alpha_\pi(f^i_l \otimes d_l) = \sum_{k \in I} \phi_k, \tag{a}$$

where $f^i_l \in \mathscr{F}^i_l$, $d_l \in \mathbf{D}_l$ and $\phi_k \in \mathrm{Hom}(\mathscr{F}^k_i, \mathbf{D}_k)$ is given by

$$\phi_k(f^k_i) = \pi_{f^k_i f^i_l}(d_l) \in \mathbf{D}_k \tag{b}$$

for $f_i^k \in \mathscr{F}_i^k$. We define a $\mathbf{C}$-linear map $\beta_\pi: \mathscr{E}'^{\mathbf{D}} \to \mathscr{E}^{\mathbf{D}}$ by

$$\beta_\pi(f_j^i \otimes f_k^j \otimes d_k) = f_j^i \otimes \pi_{f_k^j}(d_k) - (f_j^i \theta_{j,\lambda} f_k^j) \otimes d_k$$

for $f_j^i \in \mathscr{F}_j^i$, $f_k^j \in \mathscr{F}_k^j$, $d_k \in \mathbf{D}_k$.

LEMMA 2.6. *α_π and β_π are homomorphisms of $\mathscr{F}$-modules.*

The proof is immediate.

LEMMA 2.7. *$\mathscr{E}'^{\mathbf{D}} \xrightarrow{\beta_\pi} \mathscr{E}^{\mathbf{D}} \xrightarrow{\alpha_\pi} \mathscr{E}''^{\mathbf{D}}$ is a complex, that is, $\alpha_\pi \beta_\pi = 0$.*

Let $f_j^i \in \mathscr{F}_j^i$, $f_l^j \in \mathscr{F}_l^j$, $d_l \in \mathbf{D}_l$. We have

$$\alpha_\pi(\beta_\pi(f_j^i \otimes f_l^j \otimes d_l)) = \alpha_\pi(f_j^i \otimes \pi_{f_l^j}(d_l)) - \alpha_\pi((f_j^i \theta_{j,\lambda} f_l^j) \otimes d_l) = \sum_{k \in I} \phi_k,$$

where $\phi_k \in \operatorname{Hom}(\mathscr{F}_i^k, \mathbf{D}_k)$ is given by $\phi_k(f_i^k) = \pi_{f_i^k f_j^i} \pi_{f_l^j}(d_l) - \pi_{f_i^k f_j^i \theta_{j,\lambda} f_l^j}(d_l)$, which is zero by 2.4(b). The lemma is proved.

2.8. We set $\mathscr{K}^\pi = \operatorname{Ker}(\alpha_\pi)$, $\mathscr{I}^\pi = \operatorname{Im}(\beta_\pi)$. By Lemma 2.6, $\mathscr{K}^\pi$ and $\mathscr{I}^\pi$ are $\mathscr{F}$-submodules of $\mathscr{E}^{\mathbf{D}}$. By Lemma 2.7, we have

$$\mathscr{I}^\pi \subset \mathscr{K}^\pi.$$

From the definitions it is clear that α_π, β_π are compatible with the I-gradings. Hence

$$\mathscr{K}^\pi = \bigoplus_{i \in I} \mathscr{K}_i^\pi, \qquad \mathscr{I}^\pi = \bigoplus_{i \in I} \mathscr{I}_i^\pi,$$

where $\mathscr{K}_i^\pi = \mathscr{K}^\pi \cap \mathscr{E}_i^{\mathbf{D}}$, $\mathscr{I}_i^\pi = \mathscr{I}^\pi \cap \mathscr{E}_i^{\mathbf{D}}$.

2.9. We define $\mathbf{C}$-linear maps $\mathbf{p}_i: \mathbf{D}_i \to \mathscr{E}_i^{\mathbf{D}}$, $\mathbf{q}_i^\pi: \mathscr{E}_i^{\mathbf{D}} \to \mathbf{D}_i$ by

$$\mathbf{p}_i(d_i) = e_i \otimes d_i \qquad \text{for} \quad d_i \in \mathbf{D}_i,$$
$$\mathbf{q}_i(f_k^i \otimes d_k) = \pi_{f_k^i}(d_k) \qquad \text{for} \quad f_k^i \in \mathscr{F}_k^i,\ d_k \in \mathbf{D}_k.$$

Let $\zeta_i: \mathscr{E}_i''^{\mathbf{D}} \to \mathbf{D}_i$ be the linear map given by $\zeta_i(\phi) = \phi(e_i)$ for $\phi \in \operatorname{Hom}(\mathscr{F}_i^k, \mathbf{D}_k)$ with $k = i$, $\zeta_i(\phi) = 0$ for $\phi \in \operatorname{Hom}(\mathscr{F}_i^k, \mathbf{D}_k)$ with $k \neq i$. We have, from the definitions

$$\mathbf{q}_i(x) = \zeta_i(\alpha^\pi(x)) \tag{a}$$

for all $x \in \mathscr{E}_i^{\mathbf{D}}$. It follows that

$$\mathscr{K}_i^\pi \subset \mathbf{q}_i^{-1}(0).$$

We have, from the definitions

$$\pi_{f^i_j}(d_j) = \mathbf{q}_i(f^i_j \mathbf{p}_j(d_j)) \tag{b}$$

for $f^i_j \in \mathcal{F}^i_j$, $d_j \in \mathbf{D}_j$.

Lemma 2.10. *For any $x \in \mathcal{E}^{\mathbf{D}}_i$ we have $\mathbf{p}_i\mathbf{q}_i(x) = \theta_{i,\lambda} x$ modulo $\mathcal{I}^\pi_i$.*

Let $f^i_k \in \mathcal{F}^i_k$, $d_k \in \mathbf{D}_k$. We have

$$\begin{aligned}\mathbf{p}_i\mathbf{q}_i(f^i_k \otimes d_k) - \theta_{i,\lambda}(f^i_k \otimes d_k) &= e_i \otimes \pi_{f^i_k}(d_k) - (\theta_{i,\lambda} f^i_k) \otimes d_k \\ &= \beta_\pi(e_i \otimes f^i_k \otimes d_k) \in \mathcal{I}^\pi_i.\end{aligned}$$

The lemma is proved.

2.11. If $\mathbf{V}'$ is an I-graded subspace of $\mathbf{V} \in \mathcal{C}$ and $x = (x_h)_{h \in H} \in \bigoplus_{h \in H} \mathrm{Hom}(\mathbf{V}_{h'}, \mathbf{V}_{h''})$, we say that $\mathbf{V}'$ is *x-adapted* if $x_h(\mathbf{V}'_{h'}) \subset \mathbf{V}'_{h''}$ for all $h \in H$.

Let λ be as in 2.4. Following Nakajima [N2] we define $\Lambda_{\mathbf{D},\mathbf{V},\lambda}$ to be the set of all $(x, p, q) \in \mathbf{M}_{\mathbf{D},\mathbf{V}}$ (see 1.2) such that

$$\sum_{h \in H;\, h''=i} \varepsilon(h)\, x_h x_{\bar{h}} - p_i q_i - \lambda_i : \mathbf{V}_i \to \mathbf{V}_i$$

is zero for all $i \in I$.

Following [N2] we say that a triple $(x, p, q) \in \Lambda_{\mathbf{D},\mathbf{V},\lambda}$ is *stable* if it satisfies the following condition:

If S is an x-adapted I-graded subspace of $\mathbf{V}$ *containing* $\mathrm{Im}(p)$, *then* $S = \mathbf{V}$.

Let $\Lambda^s_{\mathbf{D},\mathbf{V},\lambda}$ be the set of stable triples in $\Lambda_{\mathbf{D},\mathbf{V},\lambda}$.

We can reformulate Lemma 2.10 as follows:

For $\pi \in Z^\lambda_{\mathbf{D}}$, $(\mathbf{x}^\pi, \mathbf{p}, \mathbf{q})$ belongs to $\Lambda_{\mathbf{D},\mathcal{E}^{\mathbf{D}}/\mathcal{I}^\pi,\lambda}$, where $\mathbf{x}^\pi_h \colon \mathcal{E}^{\mathbf{D}}_{h'}/\mathcal{I}^\pi_{h'} \to \mathcal{E}^{\mathbf{D}}_{h''}/\mathcal{I}^\pi_{h''}$ is multiplication by $h \in H$ in the $\mathcal{F}$-module structure of $\mathcal{E}^{\mathbf{D}}/\mathcal{I}^\pi$; $\mathbf{p}$ *has i-component induced by* $\mathbf{p}_i$; $\mathbf{q}$ *has i-component induced by* $\mathbf{q}_i$.

From the definition we see that the $\mathcal{F}$-submodule of $\mathcal{E}^{\mathbf{D}}/\mathcal{I}^\pi$ generated by $\mathrm{Im}(\mathbf{p})$ is $\mathcal{E}^{\mathbf{D}}/\mathcal{I}^\pi$ itself. Equivalently, we have

$$(\mathbf{x}^\pi, \mathbf{p}, \mathbf{q}) \in \Lambda^s_{\mathbf{D},\mathcal{E}^{\mathbf{D}}/\mathcal{I}^\pi,\lambda}. \tag{a}$$

In fact, we will see in 2.17 that $(\mathbf{x}^\pi, \mathbf{p}, \mathbf{q})$ is in some sense a *universal* stable triple.

2.12. Let $(x, p, q) \in \Lambda_{\mathbf{D},\mathbf{V},\lambda}$. We regard V as a $\mathcal{F}$-module by the requirement that, for $(h_1, h_2, ..., h_r) \in \mathcal{P}^i_j$, $v \in \mathbf{V}_{j'}$, $(h_1, h_2, ..., h_r)\, v$ is $x_{h_1} x_{h_2} \cdots x_{h_r} v \in \mathbf{V}_i$, if $j' = j$ and is 0, if $j' \neq j$. From the definitions we have

$$\theta_{i,\lambda} v_i = p_i q_i(v_i) \qquad \text{for} \quad v_i \in \mathbf{V}_i. \tag{a}$$

To (x, p, q) we attach $\pi = \{\pi_{f^i_j} \in \mathrm{Hom}(\mathbf{D}_j, \mathbf{D}_i) \mid f^i_j \in \mathscr{F}^i_j,\ i, j \in I\}$ by $\pi_{f^i_j}(d_j) = q_i(f^i_j p_j(d_j))$ for $f^i_j \in \mathscr{F}^i_j$, $d_j \in \mathbf{D}_j$. We show that

$$\pi \in Z^\lambda_{\mathbf{D}}.$$

Indeed, let $f^j_i \in \mathscr{F}^j_i$, $f^i_k \in \mathscr{F}^i_k$, $d_k \in \mathbf{D}_k$. We have

$$\pi_{f^j_i}\pi_{f^i_k}(d_k) = \pi_{f^j_i}(q_i(f^i_k p_k(d_k))) = q_j(f^j_i p_i q_i(f^i_k p_k(d_k))),$$
$$\pi_{f^j_i \theta_{i,\lambda} f^i_k}(d_k) = q_j(f^j_i \theta_{i,\lambda} f^i_k p_k(d_k)).$$

We must show that $\pi_{f^j_i}\pi_{f^i_k}(d_k) = \pi_{f^j_i \theta_{i,\lambda} f^i_k}(d_k)$ (see 2.4(b)) or, equivalently, that

$$q_j(f^j_i p_i q_i(f^i_k p_k(d_k))) = q_j(f^j_i \theta_{i,\lambda} f^i_k p_k(d_k)).$$

But this follows from $p_i q_i(f^i_k p_k(d_k)) = \theta_{i,\lambda} f^i_k p_k(d_k)$ which is a special case of (a).

2.13. Next, to $(x, p, q) \in \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$ and to $i \in I$, we attach a $\mathbf{C}$-linear map $\Phi_i\colon \mathscr{E}^{\mathbf{D}}_i \to \mathbf{V}_i$ by

$$\Phi_i(f^i_k \otimes d_k) = f^i_k p_k(d_k)$$

for $f^i_k \in \mathscr{F}^i_k$, $d_k \in \mathbf{D}_k$. Let $\Phi\colon \mathscr{E}^{\mathbf{D}} \to \mathbf{V}$ be given by $\Phi = \bigoplus_i \Phi_i$. We verify that

$$\Phi_i(\mathscr{I}^\pi_i) = 0.$$

Let $f^i_j \in \mathscr{F}^i_j$, $f^j_k \in \mathscr{F}^j_k$, $d_k \in \mathbf{D}_k$. We have

$$\begin{aligned}\Phi_i(\beta^\pi(f^i_j \otimes f^j_k \otimes d_k)) &= \Phi_i(f^i_j \otimes \pi_{f^j_k}(d_k) - (f^i_j \theta_{j,\lambda} f^j_k) \otimes d_k)\\ &= f^i_j p_j(\pi_{f^j_k}(d_k)) - f^i_j \theta_{j,\lambda} f^j_k p_k(d_k)\\ &= f^i_j p_j q_j(f^j_k p_k(d_k)) - f^i_j \theta_{j,\lambda} f^j_k p_k(d_k)\end{aligned}$$

and this is zero by 2.12(a). Thus, our claim is verified.

2.14. We show that $\Phi\colon \mathscr{E}^{\mathbf{D}} \to \mathbf{V}$ is a homomorphism of $\mathscr{F}$-modules. It is enough to show that, for $f^n_i \in \mathscr{F}^n_i$, $f^i_k \in \mathscr{F}^i_k$, $d_k \in \mathbf{D}_k$, we have

$$\Phi(f^n_i f^i_k \otimes d_k) = f^n_i \Phi(f^i_k \otimes d_k).$$

Both sides are equal to $f^n_i f^i_k p_k(d_k)$ and our claim is verified.

2.15. Let $\xi_i\colon \mathbf{V}_i \to \mathscr{E}''^{\mathbf{D}}_i$ be the linear map defined by $\xi_i(v_i)=\sum_{k\in I}\phi_k$ where $\phi_k\in\operatorname{Hom}(\mathscr{F}^k_i, \mathbf{D}_k)$ is given by $\phi_k(f^k_i)=p_k(f^k_i v_i)$ for $f^k_i\in\mathscr{F}^k_i$. From the definitions, we have

$$\alpha^\pi(x)=\xi_i(\Phi_i(x)) \tag{a}$$

for any $x\in\mathscr{E}^{\mathbf{D}}_i$. Both sides map $f^i_l\otimes d_l$ with $f^i_l\in\mathscr{F}^i_l$, $d_l\in\mathbf{D}_l$ to $\sum_{k\in I}\phi'_k$ where $\phi'_k\in\operatorname{Hom}(\mathscr{F}^k_i, \mathbf{D}_k)$ is given by $\phi'_k(f^k_i)=p_k(f^k_i f^i_l p_l(d_l))$ for $f^k_i\in\mathscr{F}^k_i$. From (a) we deduce

$$\Phi^{-1}(0)\subset\mathscr{K}^\pi. \tag{b}$$

2.16. For $d_i\in\mathbf{D}_i$, $x\in\mathscr{E}^{\mathbf{D}}_i$, we have, from the definitions,

$$p_i(d_i)=\Phi_i(\mathbf{p}_i(d_i)), \qquad q_i(\Phi_i(x))=\mathbf{q}_i(x).$$

LEMMA 2.17. *Φ: $\mathscr{E}^{\mathbf{D}}\to\mathbf{V}$ is surjective if and only if (x, p, q) is stable.*

Assume first that (x, p, q) is stable. Let $W=\operatorname{Im}(\Phi)$. We have $W=\bigoplus_{i\in I}W_i$ where $W_i=\operatorname{Im}(\Phi_i)$. Using the definition of Φ_i we see that $f^i_k(\operatorname{Im}(p_k))\subset W_i$ for any $f^i_k\in\mathscr{F}^i_k$. Hence, the $\mathscr{F}$-submodule W' of $\mathbf{V}$ generated by $\operatorname{Im}(p)$ is contained in W. Now W' is an x-adapted I-graded subspace of $\mathbf{V}$ containing $\operatorname{Im}(p)$. By the stability of (x, p, q) we have $W'=\mathbf{V}$. Hence $W=W'$ and Φ is surjective.

Conversely, assume that Φ is surjective. Let S be an x-adapted I-graded subspace of $\mathbf{V}$ containing $\operatorname{Im}(p)$. Then $f^i_k p_k(d_k)\in S$ for any $f^i_k\in\mathscr{F}^i_k$, $d_k\in\mathbf{D}_k$. Using the definition of Φ, we see that the image of Φ is contained in S. Since Φ is surjective, it follows that $S=\mathbf{V}$ and (x, p, q) is stable. The lemma is proved.

2.18. Let $\mathscr{R}_{\mathbf{D},\lambda}$ be the set of all pairs $(\pi, \mathscr{V})$ where $\pi\in Z^\lambda_{\mathbf{D}}$ and $\mathscr{V}$ is an I-graded subspace of $\mathscr{E}^{\mathbf{D}}$ such that $\mathscr{I}^\pi\subset\mathscr{V}\subset\mathscr{K}^\pi$ and such that $\mathscr{V}$ is a $\mathscr{F}$-submodule of $\mathscr{E}^{\mathbf{D}}$.

Assume that we are given $(\pi, \mathscr{V})\in\mathscr{R}_{\mathbf{D},\lambda}$. For $h\in H$, let $x_h\colon(\mathscr{E}^{\mathbf{D}}/\mathscr{V})_{h'}\to(\mathscr{E}^{\mathbf{D}}/\mathscr{V})_{h''}$ be given by multiplication by h in the $\mathscr{F}$-module structure of $\mathscr{E}^{\mathbf{D}}/\mathscr{V}$.

Let $q_i\colon(\mathscr{E}^{\mathbf{D}}/\mathscr{V})_i\to\mathbf{D}_i$ be the map induced by $\mathbf{q}_i\colon\mathscr{E}^{\mathbf{D}}\to\mathbf{D}_i$. (Recall from 2.9 that $\mathbf{q}_i$ is zero on $\mathscr{K}^\pi_i$, hence is zero on $\mathscr{V}_i$ since $\mathscr{V}_i\subset\mathscr{K}^\pi_i$.) Let $p_i\colon\mathbf{D}_i\to(\mathscr{E}^{\mathbf{D}}/\mathscr{V})_i$ be the composition of $\mathbf{p}_i\colon\mathbf{D}_i\to\mathscr{E}^{\mathbf{D}}_i$ with the canonical projection $\mathscr{E}^{\mathbf{D}}_i\to(\mathscr{E}^{\mathbf{D}}/\mathscr{V})_i$.

Let $x=(x_h)_{h\in H}$, $p=(p_i)_{i\in I}$, $q=(q_i)_{i\in I}$. From Lemma 2.10, we see that $(x, p, q)\in\Lambda_{\mathbf{D},\mathscr{E}^{\mathbf{D}}/\mathscr{V},\lambda}$. (We use the inclusion $\mathscr{I}^\pi\subset\mathscr{V}$.)

As in 2.12, (with $\mathbf{V}=\mathscr{E}^{\mathbf{D}}/\mathscr{V}$), to (x, p, q) we associate an element of $Z^\lambda_{\mathbf{D}}$. This element is in fact equal to our given π. This follows from 2.9(b).

As in 2.13, (with $\mathbf{V}=\mathscr{E}^{\mathbf{D}}/\mathscr{V}$), to (x, p, q) we associate a linear map $\Phi_i\colon \mathscr{E}_i^{\mathbf{D}} \to (\mathscr{E}^{\mathbf{D}}/\mathscr{V})_i$. This linear map is in fact equal to the obvious projection. (The verification is immediate.) In particular, $\Phi_i\colon \mathscr{E}_i^{\mathbf{D}} \to (\mathscr{E}^{\mathbf{D}}/\mathscr{V})_i$ is surjective. Hence, from Lemma 2.17, we see that $(x, p, q) \in \Lambda^s_{\mathbf{D}, \mathscr{E}^{\mathbf{D}}/\mathscr{V}, \lambda}$.

2.19. Assume that $\mathbf{D}, \lambda$ are fixed. We say that $(\mathbf{V}, x, p, q)$ is a stable quadruple if $\mathbf{V} \in \mathscr{C}$ and $(x, p, q) \in \Lambda^s_{\mathbf{D}, \mathbf{V}, \lambda}$.

Two stable quadruples $(\mathbf{V}, x, p, q)$, $(\mathbf{V}', x', p', q')$ are said to be *similar* if there exists an isomorphism $\gamma=(\gamma_i)_{i\in I}\colon \mathbf{V} \xrightarrow{\sim} \mathbf{V}'$ (in $\mathscr{C}$) such that

$$x'_h=\gamma_{h''}x_h\gamma_{h'}^{-1} \qquad \text{for all} \quad h, \qquad p'_i=\gamma_i p_i,\ q'_i=q_i\gamma_i^{-1} \qquad \text{for all} \quad i.$$

It is clear that similarity gives an equivalence relation on stable quadruples. Let $\mathscr{R}'_{\mathbf{D},\lambda}$ be the set of similarity classes of stable quadruples.

If $(\mathbf{V}, x, p, q)$ is a stable quadruple, then we can define $\pi \in Z^\lambda_\pi$ and $\Phi\colon \mathscr{E}^{\mathbf{D}} \to \mathbf{V}$ as in 2.12, 2.13 and Φ will be surjective by 2.17. Hence, if we set $\mathscr{V}=\operatorname{Ker}(\Phi)$, then $(\pi, \mathscr{V}) \in \mathscr{R}_{\mathbf{D},\lambda}$. (We use 2.14 and 2.15(b).) Clearly, $(\pi, \mathscr{V})$ depends only on the similarity class of $(\mathbf{V}, x, p, q)$. Thus, we have defined a map $\mathscr{R}'_{\mathbf{D},\lambda} \to \mathscr{R}_{\mathbf{D},\lambda}$. We now define a map in the opposite direction.

Let $(\pi, \mathscr{V}) \in \mathscr{R}_{\mathbf{D},\lambda}$. In 2.18 we have attached to $(\pi, \mathscr{V})$ a triple $(x, p, q) \in \Lambda^s_{\mathbf{D}, \mathscr{E}^{\mathbf{D}}/\mathscr{V}, \lambda}$. Associating to $(\pi, \mathscr{V})$ the similarity class of $(\mathscr{E}^{\mathbf{D}}/\mathscr{V}, x, p, q)$ we obtain a map $\mathscr{R}_{\mathbf{D},\lambda} \to \mathscr{R}'_{\mathbf{D},\lambda}$. It is clear that this is the inverse of the map $\mathscr{R}'_{\mathbf{D},\lambda} \to \mathscr{R}_{\mathbf{D},\lambda}$ considered above. Thus we have the following result.

THEOREM 2.20. *The sets $\mathscr{R}_{\mathbf{D},\lambda}$, $\mathscr{R}'_{\mathbf{D},\lambda}$ are in natural bijection.*

For $v=\sum_{i\in I} v_i i$, let $\mathscr{R}_{\mathbf{D},v,\lambda}$ be the set of all $(\pi, \mathscr{V}) \in \mathscr{R}_{\mathbf{D},\lambda}$ such that $|\mathscr{E}^{\mathbf{D}}/\mathscr{V}|=v$. Let $\mathscr{R}'_{\mathbf{D},v,\lambda}$ be the set of all equivalence classes of stable quadruples $(\mathbf{V}, x, p, q)$ with $|\mathbf{V}|=v$. The bijection in the theorem restricts to a bijection

$$\mathscr{R}_{\mathbf{D},v,\lambda} \leftrightarrow \mathscr{R}'_{\mathbf{D},v,\lambda}.$$

Note that $\mathscr{R}'_{\mathbf{D},v,\lambda}$ may be regarded as the moduli space of stable quadruples (with fixed $\mathbf{D}, v, \lambda$) and the theorem gives a way to pick up a canonical stable quadruple in any similarity class of stable quadruples (namely one of the form $(\mathscr{E}^{\mathbf{D}}/\mathscr{V}, x, p, q)$ as in 2.19).

2.21. We now assume (until the end of 2.26) that λ in 2.4 is 0.

Let $\mathbf{V} \in \mathscr{C}^0$ and let $x=(x_h)_{h\in H} \in \bigoplus_{h\in H} \operatorname{Hom}(\mathbf{V}_{h'}, \mathbf{V}_{h''})$. Following [L1, 1.7], we say that x is *nilpotent* if there exists $N \geqslant 1$ such that for any path $h_1, h_2, ..., h_r$ in (I, H) (see 1.2(b)) with $r \geqslant N$, the composition $x_{h_r}x_{h_{r-1}}\cdots x_{h_1}\colon \mathbf{V}_{h'_1} \to \mathbf{V}_{h''_r}$ is zero. If x is nilpotent, then N above may be chosen to be $\sum_i \dim \mathbf{V}_i$. (This follows from [L1, 1.8].)

Let $\Lambda^{sn}_{\mathbf{D},\mathbf{V}}$ be the set of all $(x, p, q) \in \Lambda^{s}_{\mathbf{D},\mathbf{V},0}$ such that x is nilpotent and $q=0$.

LEMMA 2.22. *Assume that $\mathbf{V} \in \mathscr{C}^0$. The following conditions for $(x, p, q) \in \Lambda^{s}_{\mathbf{D},\mathbf{V},0}$ are equivalent*:

(i) *The closure of the $G_{\mathbf{V}}$-orbit of (x, p, q) in $\mathbf{M}_{\mathbf{D},\mathbf{V}}$ contains* 0.

(ii) $(x, p, q) \in \Lambda^{sn}_{\mathbf{D},\mathbf{V}}$.

Assume that (i) holds. By the Hilbert-Mumford theorem, there exists a 1-parameter subgroup $(g_t)_{t \in \mathbf{C}^*}$ of G such that $\lim_{t\to 0} g_t(x, p, q) = (0, 0, 0)$. Let $g_t(x, p, q) = (x(t), p(t), q(t))$ for $t \in \mathbf{C}^*$. We have a direct sum decomposition $\mathbf{V} = \bigoplus_k \mathbf{W}^k$ where $g_t(v) = t^k v$ for $v \in \mathbf{W}^k$. Note that $\mathbf{W}^k$ is an I-graded subspace of $\mathbf{V}$.

We have $x_h(v) = \sum_{k'} x_{h;k,k'} v$ for $v \in \mathbf{W}^k_{h'}$, where $x_{h;k,k'}: \mathbf{W}^k_{h'} \to \mathbf{W}^{k'}_{h''}$. Then $x(t)_h(v) = \sum_{k'} x_{h;k,k'} t^{k'-k} v$ for $v \in \mathbf{W}^k_{h'}$. Since this tends to 0 for $t \to 0$, it follows that $x_{h;k,k'} = 0$ for $k' \leqslant k$. It follows that x is nilpotent and that the subspace $\bigoplus_{k>0} \mathbf{W}^k$ is x-adapted. We have $p(t)(d) = \sum_k p_k(d)\, t^k$ where $p_k: \mathbf{D} \to \mathbf{W}^k$. Since this tends to 0 for $t \to 0$ it follows that $p_k = 0$ for $k \leqslant 0$. Thus, $\bigoplus_{k>0} \mathbf{W}^k$ is both x-adapted and contains the image of p. By the stability of (x, p, q) we have $\mathbf{V} = \bigoplus_{k>0} \mathbf{W}^k$.

We have $q(t)(v) = t^{-k} q(v)$ for $v \in \mathbf{W}^k$. Since this tends to 0 for $t \to 0$ it follows that $q|_{\mathbf{W}^k} = 0$ for $k \geqslant 0$. Hence $q = 0$. Hence (ii) holds.

Conversely, assume that (ii) holds. By [L1, 1.8], we can find a sequence $\mathbf{V} = \mathbf{V}^0 \supset \mathbf{V}^1 \supset \cdots \supset \mathbf{V}^m = \mathbf{V}$ of I-graded subspaces of $\mathbf{V}$ such that $x_h(\mathbf{V}^k_{h'}) \subset \mathbf{V}^{k+1}_{h''}$ for $k \in [0, m-1]$ and all h. For $k \in [1, m]$, let $\mathbf{W}^k$ be an I-graded subspace of $\mathbf{V}^{k-1}$ complementary to $\mathbf{V}^k$. Then $\mathbf{V} = \bigoplus_{k=1}^m \mathbf{W}^k$. For $t \in \mathbf{C}^*$, let $g_t \in G_{\mathbf{V}}$ be defined by $g_t = t^k$ on $\mathbf{W}^k$. It is clear that $\lim_{t\to 0} g_t(x, p, 0) = (0, 0, 0)$. The lemma is proved.

2.23. Let $\mathscr{J}$ be the subspace of $\mathscr{F}$ spanned by the elements $f^i_j \theta_j f^j_k$ with $f^i_j \in \mathscr{F}^i_j$, $f^j_k \in \mathscr{F}^j_k$; equivalently, $\mathscr{J}$ is the two-sided ideal of $\mathscr{F}$ generated by the elements θ_j with $j \in I$. Note that

(a) *if $(x', p', q') \in \Lambda^{s}_{\mathbf{D},\mathbf{V},0}$ and the element $\pi \in Z^0_{\mathbf{D}}$ attached in* 2.12 *to (x', p', q') is* 0, *then $q' = 0$ and the subspace $\mathscr{V}$ of $\mathscr{E}^{\mathbf{D}}$ attached to $(\mathbf{V}, x', p', q')$ in* 2.19 *satisfies $\mathscr{J}\mathscr{E}^{\mathbf{D}} \subset \mathscr{V}$.*

Indeed, for $\pi = 0$ we have $\mathscr{K}^0 = \mathscr{E}^{\mathbf{D}}$ and $\mathscr{I}^0 = \mathscr{J}\mathscr{E}^{\mathbf{D}}$. Hence $\mathscr{J}\mathscr{E}^{\mathbf{D}} \subset \mathscr{V}$. By definition (2.9), we have $\mathbf{q} = 0$. Hence the quadruple $(\mathscr{E}^{\mathbf{D}}/\mathscr{V}, x, p, q)$ attached to $(0, \mathscr{V})$ in 2.18 has $q = 0$. Since this quadruple is similar to $(\mathbf{V}, x', p', q')$, we must have $q' = 0$, as claimed.

Let $(x, p, q) \in \Lambda^{sn}_{\mathbf{D},\mathbf{V}}$. The element $\pi \in Z^0_{\mathbf{D}}$ attached in 2.12 to (x, p, q) is 0 since $q = 0$. Hence the subspace $\mathscr{V}$ of $\mathscr{E}^{\mathbf{D}}$ attached to $(\mathbf{V}, x, p, q)$ in 2.19

satisfies $\mathcal{J}\mathcal{E}^{\mathbf{D}} \subset \mathcal{V}$ and any monomial m in $\mathcal{F}$ of large enough length (say, $\geqslant \dim(\mathcal{E}^{\mathbf{D}}/\mathcal{V})$) acts on $\mathcal{E}^{\mathbf{D}}/\mathcal{V}$ as zero, or equivalently $m\mathcal{E}^{\mathbf{D}} \subset \mathcal{V}$.

2.24. For $v = \bigoplus v_i$ $i \in \mathbf{N}[I]$, let $\mathcal{R}'^{n}_{\mathbf{D},v}$ be the set of all equivalence classes of stable quadruples $(\mathbf{V}, x, p, 0)$ with $|\mathbf{V}| = v$, $(x, p, 0) \in \Lambda^{sn}_{\mathbf{D},\mathbf{V}}$.

Let $\mathcal{R}^{n}_{\mathbf{D},v}$ be the set of all pairs $(\pi, \mathcal{V})$ where $\pi \in Z^0_{\mathbf{D}}$ is 0 and $\mathcal{V}$ is an I-graded subspace of $\mathcal{E}^{\mathbf{D}}$ such that $\mathcal{J}\mathcal{E}^{\mathbf{D}} \subset \mathcal{V}$, $|\mathcal{E}^{\mathbf{D}}/\mathcal{V}| = v$ and such that $\mathcal{V}$ is a $\mathcal{F}$-submodule of $\mathcal{E}^{\mathbf{D}}$ which contains $m\mathcal{E}^{\mathbf{D}}$ for any monomial m of large enough length (or equivalently, of length $\geqslant \sum_i v_i$).

The bijection in 2.20 restrics to a bijection of the subset $\mathcal{R}'^{n}_{\mathbf{D},v}$ of $\mathcal{R}'_{\mathbf{D},v,0}$ with the subset $\mathcal{R}^{n}_{\mathbf{D},v}$ of $\mathcal{R}_{\mathbf{D},v,0}$.

2.25. Let $\mathbf{P} = \mathcal{F}/\mathcal{J}$. (An associative algebra with 1.) The algebra $\mathbf{P}$ was introduced in [GP]. (In [GP] the function ε is taken to be identically one; by a change of scale this can be transformed into a function as in 2.1(a) since the graphs considered in [GP] are assumed to be trees.) In [DR], the algebra $\mathbf{P}$ is called the *preprojective algebra*. We have

$$\mathcal{J} \cap \mathcal{F}^i_j = \sum_{k \in I} \mathcal{F}^i_k \theta_k \mathcal{F}^k_j$$

since $\theta_k \in \mathcal{F}^k_k$. It follows that the direct sum decomposition $\mathcal{F} = \bigoplus_{i,j \in I} \mathcal{F}^i_j$ is compatible with the subspace $\mathcal{J}$, that is, $\mathcal{J}$ is the sum of its intersections with the summands in this decomposition. Hence, if $\mathbf{P}^i_j$ denotes the image of $\mathcal{F}^i_j$ under the canonical map $\mathcal{F} \to \mathbf{P}$, then $\mathbf{P} = \bigoplus_{i,j \in I} \mathbf{P}^i_j$.

The image in $\mathbf{P}$ of a monomial of length r in $\mathcal{F}$ is again called a monomial of length r.

Let $\bar{\mathcal{E}}^{\mathbf{D}} = \bigoplus_{i \in I} \bar{\mathcal{E}}^{\mathbf{D}}_i$ be the I-graded vector space given by

$$\bar{\mathcal{E}}^{\mathbf{D}}_i = \bigoplus_{k \in I} \mathbf{P}^i_k \otimes \mathbf{D}_k.$$

We have canonically $\bar{\mathcal{E}}^{\mathbf{D}} = \mathcal{E}^{\mathbf{D}}/\mathcal{J}\mathcal{E}^{\mathbf{D}}$. Hence the $\mathcal{F}$-module structure on $\mathcal{E}^{\mathbf{D}}$ induces a $\mathbf{P}$-module structure on $\bar{\mathcal{E}}^{\mathbf{D}}$.

THEOREM 2.26. *Let $\mathbf{D} \in \mathcal{C}^0$, $v \in \mathbf{N}[I]$. There is a canonical bijection between the set $\mathcal{R}'^{n}_{\mathbf{D},v}$ (see 2.24) and the set $\bar{\mathcal{R}}^{n}_{\mathbf{D},v}$ consisting of all I-graded subspaces $\bar{\mathcal{V}}$ of $\bar{\mathcal{E}}^{\mathbf{D}}$ such that $|\bar{\mathcal{E}}^{\mathbf{D}}/\bar{\mathcal{V}}| = v$ and such that $\bar{\mathcal{V}}$ is a $\mathbf{P}$-submodule of $\bar{\mathcal{E}}^{\mathbf{D}}$ which contains $m\bar{\mathcal{E}}^{\mathbf{D}}$ for any monomial m of large enough length (equivalently, of length $\geqslant \sum_i v_i$) in $\mathbf{P}$.*

Taking inverse under the canonical map $\mathcal{E}^{\mathbf{D}} \to \bar{\mathcal{E}}^{\mathbf{D}}$ clearly defines a bijection of $\bar{\mathcal{R}}^{n}_{\mathbf{D},v}$ onto $\mathcal{R}^{n}_{\mathbf{D},v}$. We compose this bijection with the bijection between $\mathcal{R}^{n}_{\mathbf{D},v}$ and $\mathcal{R}'^{n}_{\mathbf{D},v}$ in 2.24. This gives the desired bijection.

In the same way, using 2.24(a), we obtain a bijection between the sets (a), (b) below:

(a) the set of similarity classes of stable quadruples $(\mathbf{V}, x, p, q)$ with $|\mathbf{V}| = v$, $(x, p, q) \in \Lambda^s_{\mathbf{D}, \mathbf{V}, 0}$ such that the associated element $\pi \in Z^0_{\mathbf{D}}$ is 0;

(b) the set of all I-graded subspaces $\bar{\mathscr{V}}$ of $\bar{\mathscr{E}}^{\mathbf{D}}$ such that $\bar{\mathscr{V}}$ is a $\mathbf{P}$-submodule of $\bar{\mathscr{E}}^{\mathbf{D}}$ and $|\bar{\mathscr{E}}^{\mathbf{D}}/\bar{\mathscr{V}}| = v$.

2.27. The definition of stability that we used in 2.11 is dual to that used in [N2]. More precisely, for $\mathbf{V} \in \mathscr{C}^0$, let $\Lambda^{*s}_{\mathbf{D}, \mathbf{V}, \lambda}$ be the set of all $(x, p, q) \in \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$ such that the following condition is satisfied.

If S is an x-adapted I-graded subspace of $\mathbf{V}$ *contained in* $\mathrm{Ker}(q)$, *then* $S = 0$.

It is clear that, for $(x, p, q) \in \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$ we have $(x, p, q) \in \Lambda^{*s}_{\mathbf{D}, \mathbf{V}, \lambda}$ if and only if $({}^t\bar{x}, {}^tq, {}^tp) \in \Lambda^s_{\mathbf{D}^*, \mathbf{V}^*, \lambda}$. Here $\mathbf{D}^*$, $\mathbf{V}^*$ are the dual spaces of $\mathbf{D}$, $\mathbf{V}$ and ${}^t\bar{x}_h$ is the transpose of $x_{\bar{h}}$.

For future reference, we note the following lemmas.

Lemma 2.28. *Assume that (x, p, q), (x', p', q') are two points of $\Lambda^s_{\mathbf{D}, \mathbf{V}, \lambda} \cap \Lambda^{*s}_{\mathbf{D}, \mathbf{V}, \lambda}$ such that the associated elements π, π' in $Z^\lambda_{\mathbf{D}}$ are the same. Then (x, p, q), (x', p', q') are in the same $G_{\mathbf{V}}$-orbit.*

Let $\mathscr{V}$ (resp. $\mathscr{V}'$) be the subspace of $\mathscr{E}^{\mathbf{D}}$ corresponding to $(\mathbf{V}, x, p, q)$ (resp. $(\mathbf{V}, x', p', q')$) as in 2.19.

Under the canonical isomorphism $\mathscr{E}^{\mathbf{D}}/\mathscr{V} \xrightarrow{\sim} \mathbf{V}$, the subspace $\mathscr{K}^\pi/\mathscr{V}$ of $\mathscr{E}^{\mathbf{D}}/\mathscr{V}$ corresponds to an I-graded, x-adapted subspace of $\mathbf{V}$ contained in $\mathrm{Ker}(q)$. (See 2.15(b).) This subspace must be zero, since $(x, p, q) \in \Lambda^{*s}_{\mathbf{D}, \mathbf{V}, 0}$. It follows that $\mathscr{K}^\pi/\mathscr{V} = 0$ hence $\mathscr{K}^\pi = \mathscr{V}$. Similarly, we have $\mathscr{K}^{\pi'} = \mathscr{V}'$. Since $\pi = \pi'$, it follows that $\mathscr{V} = \mathscr{V}'$. Using 2.20, we deduce that the stable quadruples $(\mathbf{V}, x, p, q)$, $(\mathbf{V}, x', p', q')$ are similar (see 2.19). Hence (x, p, q), (x', p', q') are in the same $G_{\mathbf{V}}$-orbit. The lemma is proved.

Lemma 2.29. *If the $G_{\mathbf{V}}$-orbit of $(x, p, q) \in \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$ is closed in $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$ and (x, p, q) has trivial isotropy group in $G_{\mathbf{V}}$, then $(x, p, q) \in \Lambda^{*s}_{\mathbf{D}, \mathbf{V}, \lambda} \cap \Lambda^s_{\mathbf{D}, \mathbf{V}, \lambda}$.*

The inclusion $(x, p, q) \in \Lambda^{*s}_{\mathbf{D}, \mathbf{V}, \lambda}$ is proved in [N2, 3.24] assuming $\lambda = 0$ but, the same proof applies to any λ; applying this to $({}^t\bar{x}, {}^tq, {}^tp)$ which still satisfies the assumptions of the lemma, we obtain $(x, p, q) \in \Lambda^s_{\mathbf{D}, \mathbf{V}, \lambda}$.

One can show that the converse of the lemma holds; namely, if $(x, p, q) \in \Lambda^{*s}_{\mathbf{D}, \mathbf{V}, \lambda} \cap \Lambda^s_{\mathbf{D}, \mathbf{V}, \lambda}$, then the $G_{\mathbf{V}}$-orbit of (x, p, q) is closed in $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$ and (x, p, q) has trivial isotropy group in $G_{\mathbf{V}}$. We will not need this result in the sequel.

LEMMA 2.30. *Assume that the $G_{\mathbf{V}}$-orbit of $(x, p, q) \in \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$ is closed in $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$. Let* $\mathbf{W}$ *be the intersection of all x-adapted I-graded subspaces of* $\mathbf{V}$ *that contain* $\mathrm{Im}(p)$. *Then*:

(a) *the $G_{\mathbf{W}}$-orbit of $(x|_{\mathbf{W}}, p, q|_{\mathbf{W}})$ in $\Lambda_{\mathbf{D}, \mathbf{W}, \lambda}$ is closed in $\Lambda_{\mathbf{D}, \mathbf{W}, \lambda}$*;

(b) $(x|_{\mathbf{W}}, p, q|_{\mathbf{W}}) \in \Lambda^{s}_{\mathbf{D}, \mathbf{W}, \lambda} \cap \Lambda^{*s}_{\mathbf{D}, \mathbf{W}, \lambda}$;

(c) *there exists an I-graded subspace* $\mathbf{T}$ *of* $\mathbf{V}$ *complementary to* $\mathbf{W}$ *such that* $\mathbf{T}$ *is x-adapted, $q|_{\mathbf{T}} = 0$ and such that the $G_{\mathbf{T}}$-orbit of $(x|_{\mathbf{T}}, 0, 0)$ is closed in $\Lambda_{0, \mathbf{T}, \lambda}$.*

We choose an I-graded subspace $\mathbf{T}$ of $\mathbf{V}$ complementary to $\mathbf{W}$. Let $(\mu_t)_{t \in \mathbf{C}^*}$ be a one parameter group in $G_{\mathbf{V}}$ defined by $\mu_t = 1$ on $\mathbf{W}$ and $\mu_t = t^{-1}$ on $\mathbf{T}$. It is clear that $\lim_{t \to 0} \mu_t(x, p, q) = (x', p, q') \in \Lambda_{\mathbf{V}, \mathbf{D}, \lambda}$ where both $\mathbf{W}, \mathbf{T}$ are x'-adapted, $q|_{\mathbf{T}} = 0$ and $(x'|_{\mathbf{W}}, p, q'|_{\mathbf{W}}) = (x|_{\mathbf{W}}, p, q|_{\mathbf{W}})$.

Since (x', p, q') is in the closure of $G_{\mathbf{V}}(x, p, q)$ which is closed by assumption, we have $(x', p, q') \in G_{\mathbf{V}}(x, p, q)$. Hence it suffices to prove the existence part of (a) for (x', p, q') instead of (x, p, q). Note that the intersection of all x'-adapted I-graded subspaces of $\mathbf{V}$ that contain $\mathrm{Im}(p)$ is just $\mathbf{W}$. We set $(x'|_{\mathbf{W}}, p, q'|_{\mathbf{W}}) = (x_1, p_1, q_1)$, $x'|_{\mathbf{T}} = x_2$. It is clear that $(x_1, p_1, q_1) \in \Lambda^{s}_{\mathbf{D}, \mathbf{W}, \lambda}$ and $(x_2, 0, 0) \in \Lambda_{0, \mathbf{T}, \lambda}$.

Assume now that $(x'_1, p'_1, q'_1) \in \Lambda_{\mathbf{D}, \mathbf{W}, \lambda}$ is in the closure of the $G_{\mathbf{W}}$-orbit of (x_1, p_1, q_1) and $(x'_2, 0, 0) \in \Lambda_{0, \mathbf{T}, \lambda}$ is in the closure of the $G_{\mathbf{T}}$-orbit of $(x_2, 0, 0)$.

Then $(x'_1 \oplus x'_2, p'_1 \oplus 0, q'_1 \oplus 0)$ is in the closure of the $G_{\mathbf{V}}$-orbit of $(x_1 \oplus x_2, p_1 \oplus 0, q_1 \oplus 0)$ in $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$ hence in the orbit itself. Hence we can find $g \in G_{\mathbf{V}}$ such that

$$g(x_1 \oplus x_2, p_1 \oplus 0, q_1 \oplus 0) = (x'_1 \oplus x'_2, p'_1 \oplus 0, q'_1 \oplus 0).$$

We can write

$$g(v_1 + v_2) = g^{11}(v_1) + g^{12}(v_2) + g^{21}(v_1) + g^{22}(v_2)$$

for $v_1 \in \mathbf{W}$, $v_2 \in \mathbf{T}$, where

$$g^{11}\colon \mathbf{W} \to \mathbf{W}, \qquad g^{12}\colon \mathbf{T} \to \mathbf{W}, \qquad g^{21}\colon \mathbf{W} \to \mathbf{T}, \qquad g^{22}\colon \mathbf{T} \to \mathbf{T}.$$

We have

$$\begin{aligned} g^{11}_{h''}(x_1)_h &= (x'_1)_h\, g^{11}_{h'}, & g^{12}_{h''}(x_2)_h &= (x'_1)_h\, g^{12}_{h'},\\ g^{21}_{h''}(x_1)_h &= (x'_2)_h\, g^{21}_{h'}, & g^{22}_{h''}(x_2)_h &= (x'_2)_h\, g^{22}_{h'},\\ g^{11} p_1 &= p'_1, & g^{21} p_1 &= 0,\\ q'_1 g^{11} &= q_1, & q'_1 g^{12} &= 0. \end{aligned}$$

Let $\mathbf{W}' = \mathrm{Ker}(g^{21}\colon \mathbf{W} \to \mathbf{T}) \subset \mathbf{W}$. The equations above show that $\mathbf{W}'$ is x_1-adapted, $\mathrm{Im}(p_1) \subset \mathbf{W}'$. Since $(x_1, p_1, q_1) \in \Lambda^s_{\mathbf{D}, \mathbf{W}, \lambda}$, it follows that $\mathbf{W}' = \mathbf{W}$. Hence $g^{21} = 0$. This forces g^{11} and g^{22} to be isomorphisms. We see that

$$g^{11}(x_1, p_1, q_1) = (x'_1, p'_1, q'_1) \qquad \text{and} \qquad g^{22}(x_2, 0, 0) = (x'_2, 0, 0).$$

Thus, (x'_1, p'_1, q'_1) is in the $G_{\mathbf{W}}$-orbit of (x_1, p_1, q_1) and $(x'_2, 0, 0)$ is in the $G_{\mathbf{T}}$-orbit of $(x_2, 0, 0)$. Thus, (a), (c) are proved.

We prove (b). Since $(x_1, p_1, q_1) \in \Lambda^s_{\mathbf{D}, \mathbf{W}, \lambda}$, the isotropy group of (x_1, p_1, q_1) in $G_{\mathbf{W}}$ is trivial. (We argue as in [N2, 3.10]. If g stabilizes (x_1, p_1, q_1), then $\mathrm{Ker}(g-1)$ is an I-graded, x_1-adapted subspace of $\mathbf{W}$ containing $\mathrm{Im}(p)$ hence it is $\mathbf{W}$. Hence $g = 1$.) Thus the assumptions of Lemma 2.29 are satisfied for $(\mathbf{W}, x_1, p_1, q_1)$ instead of $(\mathbf{V}, x, p, q)$. Using that lemma, we conclude that $(x_1, p_1, q_1) \in \Lambda^s_{\mathbf{D}, \mathbf{W}, \lambda} \cap \Lambda^{*s}_{\mathbf{D}, \mathbf{W}, \lambda}$. The lemma is proved.

One can show that the space $\mathbf{T}$ in (c) is uniquely determined. We will not need this result in the sequel.

3. ORIENTATIONS

3.1. In this section we shall take λ in 2.4 to be zero.

Let $\mathcal{O}$ be the set of subsets $\omega \subset H$ such that $\omega \cup \bar{\omega} = H$, $\omega \cap \bar{\omega} = \emptyset$.

Let $\omega \in \mathcal{O}$. Let ${}_{\omega}\mathcal{F}$ be the subspace of $\mathcal{F}$ spanned by monomials which involve no $h \in \bar{\omega}$. Then ${}_{\omega}\mathcal{F}$ is a subalgebra of $\mathcal{F}$ containing the elements e_i, $(i \in I)$.

Let ${}_{\omega}\mathfrak{C}$ be the category of ${}_{\omega}\mathcal{F}$-modules that have finite dimension over $\mathbf{C}$. Any $\mathcal{M} \in {}_{\omega}\mathfrak{C}$ can be regarded as an object of $\mathcal{C}^0$ by $\mathcal{M} = \bigoplus_{i \in I} \mathcal{M}_i$ where $\mathcal{M}_i = \{x \in \mathcal{M} \mid e_i x = x\}$.

3.2. Given $s \geqslant 0$, let ${}^{s}_{\omega}\mathcal{P}^i_j$ be the set of all sequences $(h_1, h_2, ..., h_r) \in \mathcal{P}^i_j$ which contain exactly s terms in $\bar{\omega}$. Let ${}^{s}_{\omega}\mathcal{F}^{\,i}_{\,j}$ be the subspace of $\mathcal{F}^{\,i}_{\,j}$ spanned by ${}^{s}_{\omega}\mathcal{P}^i_j$. We have

$$\mathcal{J} \cap {}^{s}_{\omega}\mathcal{F}^{\,i}_{\,j} = \sum_{s' + s'' + 1 = s;\, k \in I} ({}^{s'}_{\omega}\mathcal{F}^{\,i}_{\,k})\, \theta_k ({}^{s''}_{\omega}\mathcal{F}^{\,k}_{\,j})$$

since $\theta_k \in {}^{1}_{\omega}\mathcal{F}^{\,k}_{\,k}$. It follows that the direct sum decomposition $\mathcal{F} = \bigoplus_{i, j \in I;\, s \geqslant 0} ({}^{s}_{\omega}\mathcal{F}^{\,i}_{\,j})$ is compatible with the subspace $\mathcal{J}$, that is, $\mathcal{J}$ is the sum of its intersections with the summands in this decomposition. Hence, if ${}^{s}_{\omega}\mathbf{P}^i_j$ denotes the image of ${}^{s}_{\omega}\mathcal{F}^{\,i}_{\,j}$ under the canonical map $\mathcal{F} \to \mathbf{P}$, then we have $\mathbf{P}^i_j = \bigoplus_{s \geqslant 0} ({}^{s}_{\omega}\mathbf{P}^i_j)$.

For any $k \in I$ and $s \in \mathbf{N}$, we set ${}^{s}_{\omega}\mathbf{P}_k = \bigoplus_{i \in I} ({}^{s}_{\omega}\mathbf{P}^i_k)$.

LEMMA 3.3. *Assume that* (I, ω) *has no cycles other than the empty ones. Then*

(a) $\dim({}^{s}_{\omega}\mathcal{F}^{i}_{j}) < \infty$; *hence* $\dim({}^{s}_{\omega}\mathbf{P}^{i}_{j}) < \infty$;

(b) ${}^{s}_{\omega}\mathbf{P}_{k}$ *is a left* ${}_{\omega}\mathcal{F}$*-submodule of* $\mathbf{P}$.

(*The left* ${}_{\omega}\mathcal{F}$*-module structure of* $\mathbf{P}$ *is given by restriction of scalars via the composition* ${}_{\omega}\mathcal{F} \to \mathcal{F} \to \mathbf{P}$).

Our assumption implies that there are only finitely many paths for (I, ω). From this (a) follows easily. The proof of (b) is immediate.

PROPOSITION 3.4. *Under the assumption of* 3.3, *we have a direct sum decomposition* $\mathbf{P} = \bigoplus_{s \in N;\, k \in I} ({}^{s}_{\omega}\mathbf{P}_{k})$ *as left* ${}_{\omega}\mathcal{F}$*-modules. Moreover, each* ${}^{s}_{\omega}\mathbf{P}_{k}$ *belongs to* ${}_{\omega}\mathfrak{C}$.

This follows immediately from Lemma 3.3.

3.5. In the remainder of this section we fix a subset $J \subset I$ and $\omega, {}'\omega \in \mathcal{O}$ such that for $h \in H$ we have

(a) $h' \in J \Rightarrow h \in \omega \cap {}'\bar{\omega}$,

(b) $h \in \omega, h' \notin J$ if and only if $h \in {}'\omega, h' \notin J$.

Moreover, we shall assume that $\varepsilon: H \to \mathbf{C}^*$ satisfies

(c) $\varepsilon(h_1) = \varepsilon(h_2)$ whenever $h'_1 = h'_2 \in J$.

Let $\mathcal{M} = \bigoplus_{i \in I} \mathcal{M}_i \in {}_{\omega}\mathfrak{C}$. Following [BGP], we attach to $\mathcal{M}$ an object $C_J\mathcal{M} \in {}_{'\omega}\mathfrak{C}$ as follows. We set $C_J\mathcal{M} = \bigoplus_{i \in I} (C_J\mathcal{M})_i$ where $C_J\mathcal{M}_i = \mathcal{M}_i$ for $i \in I - J$,

$$(C_J\mathcal{M})_j = \operatorname{Coker}\left(T_j\colon \mathcal{M}_j \to \bigoplus_{h \in H;\, h' = j} \mathcal{M}_{h''}\right)$$

for $j \in J$; here, T_j is the $\mathbf{C}$-linear map whose h-component is left multiplication by $h \in {}_{\omega}\mathcal{F}$. To specify the ${}_{'\omega}\mathcal{F}$-module structure, we must give for any $\tilde{h} \in {}'\omega$ a linear map $(C_J\mathcal{M})_{\tilde{h}'} \to (C_J\mathcal{M})_{\tilde{h}''}$. (Note that $\tilde{h}' \notin J$.) If $\tilde{h}'' \notin J$, then this linear map is just the map $\mathcal{M}_{\tilde{h}'} \to \mathcal{M}_{\tilde{h}''}$ given by the ${}_{\omega}\mathcal{F}$-module structure of $\mathcal{M}$. If $\tilde{h}'' \in J$, then this linear map is the composition

$$\mathcal{M}_{\tilde{h}'} \to \bigoplus_{h \in H;\, h' = \tilde{h}''} \mathcal{M}_{h''} \to \operatorname{Coker}(T_{\tilde{h}''})$$

(the first map is the obvious isomorphism onto the summand corresponding to $h = \bar{\tilde{h}}$; the second map is the canonical one).

Note that $\mathcal{M} \to C_J\mathcal{M}$ is naturally a functor ${}_{\omega}\mathfrak{C} \to {}_{'\omega}\mathfrak{C}$.

We say that $\mathcal{M} \in {}_{\omega}\mathfrak{C}_{\spadesuit}$ if T_j is injective for all $j \in J$.

3.6. For $j \in I$ we define a reflection $s_j\colon \mathbf{Z}[I] \to \mathbf{Z}[I]$ by $s_j(\sum v_i i) = \sum v'_i i$ where

$$v'_i = v_i \quad \text{for} \quad i \neq j \quad \text{and} \quad v'_j = -v_j + \sum_{h;\, h' = j} v_{h''}.$$

Let W be the subgroup of $\mathrm{Aut}(\mathbf{Z}[I])$ generated by $\{s_j \mid j \in I\}$.

Let $s_J = \prod_{j \in J} s_j \in W$. The factors s_j in the product commute, hence they can be multiplied in any order. We have $s_J(\sum v_i i) = \sum v'_i i$ where

$$v'_i = v_i \quad \text{for} \quad i \notin J \quad \text{and} \quad v'_j = -v_j + \sum_{h;\, h' = j} v_{h''} \quad \text{for} \quad j \in J.$$

PROPOSITION 3.7. *Let $M \in {}_\omega\mathfrak{C}$.*

(a) *If $\mathscr{M} \in {}_\omega\mathfrak{C}_\spadesuit$ then $|C_J \mathscr{M}| = s_J |\mathscr{M}|$.*

(b) *If $\mathscr{M}$ is indecomposable in ${}_\omega\mathfrak{C}$ and is in ${}_\omega\mathfrak{C}_\spadesuit$ then $C_J\mathscr{M}$ is indecomposable in ${}_{\omega'}\mathfrak{C}$.*

(c) *If $\mathscr{M}$ is indecomposable in ${}_\omega\mathfrak{C}$ and is not in ${}_\omega\mathfrak{C}_\spadesuit$ then $C_J\mathscr{M} = 0$ and $s_J|\mathscr{M}| \notin \mathbf{N}[I]$.*

This goes back to [BGP]. See also [L2, Section 2].

The following result is closely related to a result of Gelfand and Ponomarev [GP]. Note that [GP] deals only with graphs that are trees. Several results in [GP] were extended by Dlab and Ringel [DR] to general graphs but avoiding the "reflection functor" C_J. Thus the following result is not contained in either [GP] or [DR], although it is closely related to ideas in those papers.

PROPOSITION 3.8. *Let $s \in \mathbf{N}$ and let $k \in I$. Define $s' \in \mathbf{N}$ by $s' = s + \delta_{k,J}$ where $\delta_{k,J} = 1$ if $k \in J$ and $\delta_{k,J} = 0$ if $k \notin J$. Assume that (I, ω) has no cycles other than the empty ones. We have canonically $C_J({}^s_\omega\mathbf{P}_k) = {}^{s'}_{'\omega}\mathbf{P}_k$.*

This follows from the definitions, using Lemmas 3.9, 3.11 below.

LEMMA 3.9. *For any $i \in I - J$ we have*

(a) ${}^s_\omega\mathscr{P}^i_k = {}^{s'}_{'\omega}\mathscr{P}^i_k$,

(b) ${}^s_\omega\mathscr{F}^i_k = {}^{s'}_{'\omega}\mathscr{F}^i_k$, ${}^s_\omega\mathbf{P}^i_k = {}^{s'}_{'\omega}\mathbf{P}^i_k$.

Clearly, (a) implies (b).

We prove (a). Let $(h_1, h_2, ..., h_r) \in {}^s_\omega\mathscr{P}^i_k$. We have $(h_1, h_2, ..., h_r) \in {}^{s''}_\omega\mathscr{P}^i_k$ for a well defined $s'' \in \mathbf{N}$. Let S (resp. $'S$) be the set of all $n \in [1, r]$ such that $h_n \in \bar{\omega}$ (resp. $h_n \in {}'\bar{\omega}$). Let $S_0 = S \cap {}'S$. We have $S - S_0 = \{n \in [1, r] \mid h''_n \in J\}$, $'S - S_0 = \{n \in [1, r] \mid h'_n \in J\}$. Since $h''_1 = i \notin J$, we have $1 \notin S - S_0$.

Let $n \in S - S_0$. Then, $n \neq 1$ (see above) and $h'_{n-1} = h''_n \in J$. Thus, $n-1 \in {}'S - S_0$.

Let $n \in {}'S - S_0, n \neq r$. Then $h''_{n+1} = h'_n \in J$. Thus, $n+1 \in S - S_0$.

Thus, if $r \notin {}'S - S_0$, then $n \mapsto n+1$ is a bijection of ${}'S - S_0$ onto $S - S_0$. Hence $s'' = \sharp({}'S) = \sharp(S) = s$. If $r \in {}'S - S_0$, then $n \mapsto n+1$ is a bijection of $({}'S - S_0) - \{r\}$ onto $S - S_0$. Hence $s'' = \sharp({}'S) = \sharp(S) + 1 = s+1$.

Now the condition that $r \in {}'S - S_0$ is equivalent to the condition that $h'_r \in J$ hence to the condition that $k \in J$ (since $h'_r = k$). This proves (a). The lemma is proved.

3.10. For any $j \in J$ we define a diagram

$$ {}^{s}_{\omega}\mathbf{P}^{j}_{k} \xrightarrow{A} \bigoplus_{h;\, h'=j} ({}^{s}_{\omega}\mathbf{P}^{h''}_{k}) = \bigoplus_{h;\, h'=j} ({}^{s'}_{'\omega}\mathbf{P}^{h''}_{k}) \xrightarrow{B} {}^{s'}_{'\omega}\mathbf{P}^{j}_{k} \tag{a} $$

as follows. The middle equality is given by Lemma 3.9(b). The linear map A has h-component given by left multiplication by h. (We have $h \in \omega$, by 3.5(a)). The linear map B has h-component given by $\bar{h} \in {}'\omega$.

LEMMA 3.11. (a) *We have* $\mathrm{Im}(A) = \mathrm{Ker}(B)$.

(b) *B is surjective.*

We have a commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{A'} & Y' & \xrightarrow{B'} & Z' \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow \\ X & \xrightarrow{A} & Y & \xrightarrow{B} & Z \end{array}$$

Here $X \xrightarrow{A} Y \xrightarrow{B} Z$ is the diagram 3.10(a), $X' \xrightarrow{A'} Y' \xrightarrow{B'} Z'$ is the diagram

$$ {}^{s}_{\omega}\mathcal{F}^{j}_{k} \xrightarrow{A'} \bigoplus_{h;\, h'=j} ({}^{s}_{\omega}\mathcal{F}^{h''}_{k}) = \bigoplus_{h;\, h'=j} ({}^{s'}_{'\omega}\mathcal{F}^{h''}_{k}) \xrightarrow{B'} {}^{s'}_{'\omega}\mathcal{F}^{j}_{k} \tag{c} $$

where A', B' are given by the same formulas as A, B. The vertical maps a, b, c are the canonical surjections. It is clear that B' is an isomorphism. This, together with the surjectivity of c implies (b).

We now prove (a). For any $f^j_k \in {}^{s}_{\omega}\mathcal{F}^{j}_{k}$ we have from the definition and from 3.5(c) that $B'A'(f^j_k) = \pm\theta_j f^j_k \in \mathcal{J}$. Hence $BA = 0$. It remains to prove that $\mathrm{Ker}(B) \subset \mathrm{Im}(A)$. It is enough to show that $b^{-1}\,\mathrm{Ker}(B) \subset b^{-1}\,\mathrm{Im}(A)$ (since b is surjective), or equivalently that $\mathrm{Ker}(cB') \subset \mathrm{Ker}(b) + \mathrm{Im}(A')$. Since B' is an isomorphism, it is enough to show that $B'(\mathrm{Ker}(cB')) \subset B'(\mathrm{Ker}(b)) + B'(\mathrm{Im}(A'))$, or equivalently that $\mathrm{Ker}(c) \subset B'(\mathrm{Ker}(b)) + \theta_j X'$.

Now $\mathrm{Ker}(c)={}^{s'}_{'\omega}\mathcal{F}^j_k\cap\mathcal{J}$ is spanned by elements $f^j_l\theta_l f^l_k$ where $f^j_l\in{}^{s_1}_{'\omega}\mathcal{P}^j_l$, $f^l_k\in{}^{s_2}_{'\omega}\mathcal{P}^l_k$ and $s_1+s_2+1=s'$. Hence it is enough to show that

$$f^j_l\theta_l f^l_k\in B'(\mathrm{Ker}(b))+\theta_j X' \tag{d}$$

for any f^j_l, f^l_k as above.

Assume first that f^j_l has length >0. Then $f^j_l=\bar{h}f^{h''}_k$ where $h\in H$ satisfies $h'=j$ and $f^{h''}_k\in{}^{s_1}_{'\omega}\mathcal{P}^{h''}_k$. Hence $f^j_l\theta_l f^l_k=\bar{h}(f^{h''}_k\theta_l f^l_k)\in B'(\mathrm{Ker}(b))$. (Note that $f^{h''}_k\theta_l f^l_k\in\mathrm{Ker}(b)$.)

Assume next that $f^j_l=e_j$ (hence $l=j$ and we can write f^j_k instead of f^l_k). Then $f^j_l\theta_l f^l_k=\theta_j f^j_k$. We show that

$$f^j_k\in{}^s\mathcal{P}^j_k. \tag{e}$$

We have $f^j_k\in{}^{s_0}_{\omega}\mathcal{P}^j_k$ for a well-defined $s_0\in\mathbf{N}$. Consider the diagram analogous to (c) in which s is replaced by s_0, s' by s'_0 and A', B' become A'_0, B'_0. We have $B'_0A'_0(f^j_k)=\pm\theta_j f^j_k\in{}^{s'_0}_{'\omega}\mathcal{F}^j_k$. But we have also $\theta_j f^j_k\in{}^{s'}_{'\omega}\mathcal{F}^j_k$. Since $\theta_j f^j_k\neq 0$, we must have $s'_0=s'$. We have $s'=s+\delta_{k,J}$, $s'_0=s_0+\delta_{k,J}$. Hence $s=s_0$. Thus, (e) is established and we have $f^j_l\theta_l f^l_k=\theta_j f^j_k\in\theta_j X'$. Thus, (d) is established. The lemma is proved.

4. EVEN GRAPHS

4.1. In this section we assume that (I,H) (as in 1.1, 1.2) satisfies the following evenness property:

There exists a partition $I=I^1\sqcup I^{-1}$ such that for any $h\in H$, the vertices h', h'' cannot be both in I^1 or both in I^{-1}.

We fix I^1, I^{-1} as above. For $\delta=\pm1$, let $\omega^\delta=\{h\in H\mid h'\in I^\delta, h''\in I^{-\delta}\}$. Then $w^\delta\in\mathcal{O}$ (see 3.1) and $\bar{\omega}^\delta=\omega^{-\delta}$. Note that (I,ω^δ) has no cycles other than the trivial ones.

If $\mathbf{V}\in\mathcal{C}$ we define ${}^\delta\mathbf{V}=\bigoplus_{i\in I^\delta}\mathbf{V}_i$ where $\delta=\pm1$; we have $\mathbf{V}={}^1\mathbf{V}\oplus{}^{-1}\mathbf{V}$.

In this section we take ε: $H\to\mathbf{C}^*$ such that $\varepsilon(h)=1$ if $h\in\omega^1$ and $\varepsilon(h)=-1$ if $h\in\omega^{-1}$.

Now $(\omega,{}'\omega,J)=(\omega^\delta,\omega^{-\delta},I^\delta)$ satisfies 3.5(a),(b) and $J=I^\delta$ satisfies 3.5(c). Thus, the results in Section 3 are applicable to $(\omega,{}'\omega,J)=(\omega^\delta,\omega^{-\delta},I^\delta)$ where $\delta=\pm1$.

4.2. Let $\mathbf{D}\in\mathcal{C}^0$. We define inductively an object $\mathbf{D}^u=\bigoplus_{i\in I}\mathbf{D}^u_i\in\mathcal{C}^0$ for any $u\in\mathbf{Z}$ and a linear map $\phi_{h;u}$: $\mathbf{D}^u_{h'}\to\mathbf{D}^{u+1}_{h''}$ for any $u\in\mathbf{Z}$ and any $h\in H$ as follows.

For $u<0$, we set $\mathbf{D}^u=0$. For $u=0$, we set $\mathbf{D}^0=\mathbf{D}$. For $u<0$, the map $\phi_{h;u}$ (for $h\in H$) is the zero map. Assume that for some integer $u_0\geqslant0$, $\mathbf{D}^u$ is already defined for $u \leqslant u_0$ and $\phi_{h;u}\colon \mathbf{D}_{h'}^u \to \mathbf{D}_{h''}^{u+1}$ is already defined for $u < u_0$ and $h \in H$. We define

$$\mathbf{D}_i^{u_0+1} = \operatorname{Coker}\left(\mathbf{D}_i^{u_0-1} \xrightarrow{\tau} \bigoplus_{h \in H;\, h'=i} \mathbf{D}_{h''}^{u_0}\right),$$

where τ is the linear map whose h-component is $\phi_{h;u_0-1}$. For $\tilde{h} \in H$ we define $\phi_{\tilde{h};u_0}\colon \mathbf{D}_{\tilde{h}'}^{u_0} \to \mathbf{D}_{\tilde{h}''}^{u_0+1}$ as the composition

$$\mathbf{D}_{\tilde{h}'}^{u_0} \to \bigoplus_{h \in H;\, h'=\tilde{h}''} \mathbf{D}_{h''}^{u_0} \to \mathbf{D}_{\tilde{h}''}^{u_0+1}$$

(the first map is the identity isomorphism of $\mathbf{D}_{\tilde{h}'}^{u_0}$ onto the direct summand corresponding to $h = \bar{\tilde{h}}$; the second map is the canonical one). This completes the inductive definition of $\mathbf{D}_i^u$, $\phi_{h;u}$.

4.3. Let $\mathbf{D}^\dagger = \bigoplus_{u \in \mathbf{Z}} \mathbf{D}^u$. We can regard $\mathbf{D}^\dagger$ as a left $\mathscr{F}$-module as follows: e_i acts as the identity map on $\mathbf{D}_i^u$ and as 0 on $\mathbf{D}_j^u$ where $j \neq i$; the action of $h \in H$ on $x \in \mathbf{D}_i^u$ is $hx = \phi_{h;u} x \in \mathbf{D}_{h''}^{u+1}$ if $h' = i$ and $hx = 0$ if $h' \neq i$. It is clear that this $\mathscr{F}$-module structure factors through a $\mathbf{P}$-module structure.

By restriction of scalars, this $\mathbf{P}$-module can be regarded as ${}_\delta\mathscr{F}$-module, for $\delta = \pm 1$. (We shall write ${}_\delta\mathscr{F}$, ${}_\delta\mathfrak{C}$ instead of ${}_{\omega^\delta}\mathscr{F}$, ${}_{\omega^\delta}\mathfrak{C}$.) For $u \in \mathbf{Z}$, $\delta = \pm 1$ we set

$$\mathbf{D}[u, \delta] = {}^\delta\mathbf{D}^{u-1} \oplus {}^{-\delta}\mathbf{D}^u.$$

Clearly, $\mathbf{D}[u, \delta]$ is a ${}_\delta\mathscr{F}$-submodule of $\mathbf{D}^\dagger$. It is a finite dimensional $\mathbf{C}$-vector space, hence it is in ${}_\delta\mathfrak{C}$. We have $D^\dagger = \bigoplus_{u \in \mathbf{Z}} \mathbf{D}[u, \delta]$ as a ${}_\delta\mathscr{F}$-module for $\delta = \pm 1$. From the definitions, we have

$$C_\delta(\mathbf{D}[u, \delta]) = \mathbf{D}[u+1, -\delta], \tag{a}$$

where we write C_δ instead of C_{I^δ}. (See 3.5.)

4.4. From the definitions, we see that the $\mathbf{P}$-module $\mathbf{D}^\dagger$ is generated by $\mathbf{D}^0 = \mathbf{D}$. Hence there is a unique (surjective) homomorphism of $\mathbf{P}$-modules

$$\sigma\colon \bar{\mathscr{E}}^{\mathbf{D}} = \bigoplus_{i,k \in I} \mathbf{P}_k^i \otimes \mathbf{D}_k \to \mathbf{D}^\dagger$$

which takes $e_k \otimes d_k$ to d_k for any $k \in I$, $d_k \in \mathbf{D}_k$.

THEOREM 4.5. *σ is an isomorphism of $\mathbf{P}$-modules.*

If $\mathbf{D}=\mathbf{D}'\oplus\mathbf{D}''$ where $\mathbf{D}', \mathbf{D}''\in\mathscr{C}^0$ and if the theorem is true when $\mathbf{D}$ is replaced by $\mathbf{D}'$ or by $\mathbf{D}''$, then clearly, the theorem is true for $\mathbf{D}$. Hence to prove the theorem, we may assume that for some $k\in I$ we have

(a) $\mathbf{D}_k=\mathbf{C}$ and $\mathbf{D}_i=0$ for all $i\in I-\{k\}$.

The assumption (a) will be in force until the end of 4.10.

Let δ be such that $k\in I^\delta$. We set $\mathbf{D}[u]=\mathbf{D}[u,(-1)^{u-1}\delta]$; we have $\mathbf{D}[u,(-1)^u\delta]=0$.

4.6. For $u\geqslant 0$ we define $v_u\in\mathbf{Z}[I]$ inductively by $v_0=k$ and $v_u=s_{I(-1)^u\delta}\,v_{u-1}$ for $u\geqslant 1$. (Notation of 3.6.) We define $N\in[0,\infty]$ as follows.

If $v_u\in\mathbf{N}[I]$ for all $u\geqslant 1$, we set $N=\infty$. If $v_u\notin\mathbf{N}[I]$ for some $u\geqslant 1$ we define $N\in\mathbf{N}$ by the condition

$$v_0\in\mathbf{N}[I],\ v_1\in\mathbf{N}[I],\ ...,\ v_N\in\mathbf{N}[I],\ v_{N+1}\notin\mathbf{N}[I].$$

LEMMA 4.7. (a) *Assume that $N<\infty$. Then, for $u\in[0,N]$, $\mathbf{D}[u]$ is an indecomposable object of ${}_{(-1)^{u-1}\delta}\mathfrak{C}$ and $|\mathbf{D}[u]|=v_u$. If $u\in[0,N-1]$, then $\mathbf{D}[u]\in{}_{(-1)^{u-1}\delta}\mathfrak{C}_\spadesuit$, while if $u=N$, then $\mathbf{D}[u]\notin{}_{(-1)^{u-1}\delta}\mathfrak{C}_\spadesuit$ and for $u>N$ we have $\mathbf{D}[u]=0$.*

(b) *Assume that $N=\infty$. Then, for $u\geqslant 0$, $\mathbf{D}[u]$ is an indecomposable object of ${}_{(-1)^{u-1}\delta}\mathfrak{C}$ and $|\mathbf{D}[u]|=v_u$. For all $u\geqslant 0$, we have $\mathbf{D}[u]\in{}_{(-1)^{u-1}\delta}\mathfrak{C}_\spadesuit$.*

We prove the following statement.

Assume that $u\geqslant 1$ and that $\mathbf{D}[u-1]$ is an indecomposable object of ${}_{(-1)^{u-2}\delta}\mathfrak{C}$; assume also that $\mathbf{D}[u-1]\in{}_{(-1)^{u-2}\delta}\mathfrak{C}_\spadesuit$ and that $|\mathbf{D}[u-1]|=v_{u-1}$. Then

(c) *$\mathbf{D}[u]$ is an indecomposable object of ${}_{(-1)^{u-1}\delta}\mathfrak{C}$ and $|\mathbf{D}[u]|=v_u$;*

(d) *if furthermore, $v_{u+1}\in\mathbf{N}[I]$, then $\mathbf{D}[u]\in{}_{(-1)^{u-1}\delta}\mathfrak{C}_\spadesuit$;*

(e) *if on the other hand, $v_{u+1}\notin\mathbf{N}[I]$, then $\mathbf{D}[u]\notin{}_{(-1)^{u-1}\delta}\mathfrak{C}_\spadesuit$ and $\mathbf{D}[u+1]=0$.*

Since $\mathbf{D}[u-1]\in{}_{(-1)^{u-2}\delta}\mathfrak{C}_\spadesuit$, we have

$$|\mathbf{D}[u]|=|C_{(-1)^u\delta}(\mathbf{D}[u-1])|=s_{I(-1)^u\delta}v_{u-1}=v_u$$

(see 3.7(a)). In particular, $C_{(-1)^u\delta}(\mathbf{D}[u-1])\neq 0$. Since $\mathbf{D}[u-1]$ is indecomposable, we deduce that $\mathbf{D}[u]=C_{(-1)^u\delta}(\mathbf{D}[u-1])$ is indecomposable (see 3.7(a)). This proves (c).

Assume now that $\mathbf{D}[u]\notin{}_{(-1)^{u-1}\delta}\mathfrak{C}_\spadesuit$; since $\mathbf{D}[u]$ is indecomposable, from 3.7(c) we deduce $v_{u+1}=s_{I(-1)^{u+1}\delta}v_u\notin\mathbf{N}[I]$. This proves (d).

Next assume that $\mathbf{D}[u]\in{}_{(-1)^{u-1}\delta}\mathfrak{C}_\spadesuit$. Then, by the proof of (c) with u replaced by $u+1$, we would deduce that $|\mathbf{D}[u+1]|=v_{u+1}$; hence $v_{u+1}\in\mathbf{N}[I]$.

Thus, under the assumption of (e), we must have $D[u]\notin {}_{(-1)^{u-1}\delta}\mathfrak{C}_{\spadesuit}$ and then 3.7(c) gives $\mathbf{D}[u+1]=C_{(-1)^{u+1}\delta}(\mathbf{D}[u])=0$. Thus (e) holds.

Clearly, the lemma follows from (c), (d), (e) by induction.

LEMMA 4.8. *Assume that (I, H) is connected and of finite type, that is, W (see 3.6) is finite. Let $\mathbf{c}$ be the Coxeter number of W.*

(a) *If $u\leqslant \mathbf{c}-1$, then* $\sum_{i\in I^{(-1)^u\delta}}\dim \mathbf{D}_i^u i+\sum_{i\in I^{(-1)^{u-1}\delta}}\dim \mathbf{D}_i^{u-1} i=v_u$.

(b) *If $u\geqslant \mathbf{c}-1$ and $i\in I$, we have $\mathbf{D}_i^u=0$.*

In this case, it is well known that N in 4.6 is equal to $\mathbf{c}-1$ and that

(c) $v_{c-1}=k'\in I^{(-1)^{\mathbf{c}}\delta}$

where k' is the image of k under the opposition involution of I. Hence (a) follows from 4.7. Now (b) also follows from 4.7, except for the statement that

$$\mathbf{D}_i^{\mathbf{c}-1}=0 \qquad \text{for} \quad i\in I^{(-1)^{\mathbf{c}-1}\delta}$$

which follows from (a) with $u=\mathbf{c}-1$ and from (c).

Proof of Theorem 4.5. We write ${}^{s}_{\delta'}\mathbf{P}^i_k$ instead of ${}^{s}_{\omega\delta'}\mathbf{P}^i_k$. We also write ${}^{s}_{\delta'}\mathbf{P}_k$ instead of ${}^{s}_{\omega\delta'}\mathbf{P}_k$. From the definitions, we see that σ carries

${}^{s}_{-\delta}\mathbf{P}^i_k$ into $\mathbf{D}^{2s}_i$ if $i\in I^{\delta}$; ${}^{s}_{-\delta}\mathbf{P}^i_k$ into $\mathbf{D}^{2s-1}_i$ if $i\in I^{-\delta}$;
${}^{s}_{\delta}\mathbf{P}^i_k$ into $\mathbf{D}^{2s+1}_i$ if $i\in I^{\delta}$; ${}^{s}_{\delta}\mathbf{P}^i_k$ into $\mathbf{D}^{2s}_i$ if $i\in I^{\delta}$.

Hence, σ carries ${}^{s}_{\delta}\mathbf{P}_k$ into $\mathbf{D}[2s+1]$ and ${}^{s}_{-\delta}\mathbf{P}_k$ into $\mathbf{D}[2s]$. Since σ is surjective and

$$\mathbf{P}_k=\bigoplus_{s\geqslant 0}({}^{s}_{\delta}\mathbf{P}_k)=\bigoplus_{s\geqslant 0}({}^{s}_{-\delta}\mathbf{P}_k),$$

$$\mathbf{D}^{\dagger}=\bigoplus_{s\geqslant 0}\mathbf{D}[2s+1]=\bigoplus_{s\geqslant 0}\mathbf{D}[2s],$$

it follows that σ restricts to surjective maps

(a) $\quad {}^{s}_{\delta}\mathbf{P}_k\to\mathbf{D}[2s+1], \qquad {}^{s}_{-\delta}\mathbf{P}_k\to\mathbf{D}[2s].$

From the formulas

$$C_{-\delta}(\mathbf{D}[2s])=\mathbf{D}[2s+1], \qquad C_{\delta}(\mathbf{D}[2s+1])=\mathbf{D}[2s+2],$$

$$C_{-\delta}({}^{s}_{-\delta}\mathbf{P}_k)={}^{s}_{\delta}\mathbf{P}_k, \qquad C_{\delta}({}^{s}_{\delta}\mathbf{P}_k)={}^{s+1}_{-\delta}\mathbf{P}_k$$

(see 3.8, 4.3(a)) we see by induction on $s \geqslant 0$ that

$$ {}^{s}_{\delta}\mathbf{P}_k \cong \mathbf{D}[2s+1] \qquad \text{as} \quad {}_{\delta}\mathscr{F}\text{-modules}, $$

$$ {}^{s}_{-\delta}\mathbf{P}_k \cong \mathbf{D}[2s] \qquad \text{as} \quad {}_{-\delta}\mathscr{F}\text{-modules}. $$

(To start the induction, note that ${}^{0}_{-\delta}\mathbf{P}_k$, $\mathbf{D}[0]$ are both one dimensional, concentrated in degree k.)

Hence the surjective maps (a) must be isomorphisms. They are linear maps between vector spaces of the same (finite) dimension. Hence σ itself must be an isomorphism. The theorem is proved.

4.10. The previous argument shows also that σ restricts to isomorphisms

$$ {}^{s}_{-\delta}\mathbf{P}^i_k \cong \mathbf{D}^{2s}_i \quad \text{if} \quad i \in I^{\delta}; \qquad {}^{s}_{-\delta}\mathbf{P}^i_k \cong \mathbf{D}^{2s-1}_i \quad \text{if} \quad i \in I^{-\delta}; $$

$$ {}^{s}_{\delta}\mathbf{P}^i_k \cong \mathbf{D}^{2s+1}_i \quad \text{if} \quad i \in I^{-\delta}; \qquad {}^{s}_{\delta}\mathbf{P}^i_k \cong \mathbf{D}^{2s}_i \quad \text{if} \quad i \in I^{\delta}. $$

COROLLARY 4.11. (Compare [L1, 14.2(1)].) *Assume that (I, H) is connected, of finite type. Let* $\mathbf{c}$ *be as in* 4.8. *If $m \in \mathscr{F}^i_k$ is a monomial of length $u \geqslant \mathbf{c}-1$ in $\mathscr{F}$ then $m \in \mathscr{J}$.*

We must show that the image $\bar{m}$ of m in $\mathbf{P}^i_k$ is zero. By 4.8 (applied to $\mathbf{D}$ as in 4.5(a)), it suffices to show that $\bar{m}\mathbf{D}^0_k = 0$ in $\mathbf{D}^\dagger$. But $\bar{m}\mathbf{D}^0_k \in \mathbf{D}^u_i$ and this is zero, by 4.8.

4.12. In the setup of 4.11, note that there are only finitely many monomials of length $\leqslant \mathbf{c}-2$ in $\mathscr{F}$. Hence 4.11 implies that $\mathbf{P}$ is finite dimensional over $\mathbf{C}$ (a result due to Gelfand and Ponomarev [GP]).

4.13. Let $\mathbf{D} \in \mathscr{C}^0$ and let $v \in \mathbf{N}[I]$. Under the identification $\bar{\mathscr{E}}^{\mathbf{D}} = \mathbf{D}^\dagger$ as $\mathbf{P}$-modules (see 4.5), the variety $\bar{\mathscr{R}}^n_{\mathbf{D}, v}$ in 2.26 is identified with the algebraic variety consisting of all $\mathbf{P}$-submodules $\bar{\mathscr{V}}$ of $\mathbf{D}^\dagger$ which contain $\mathbf{D}^u_i$ for $i \in I$ and large enough u and satisfy $|\mathbf{D}^\dagger/\bar{\mathscr{V}}| = v$.

5. THE MAP ϑ

5.1. Let λ be as in 2.4. In this section we assume that (I, H) is connected and of finite type. Let $\mathbf{c}$ be as in 4.8. The assumption of 4.1 is now satisfied; we shall take ε as in 4.1.

5.2. Let $\mathbf{D} \in \mathscr{C}$. For any $f \in \mathscr{F}^i_j$ and any linear form χ on $\mathrm{Hom}(\mathbf{D}_j, \mathbf{D}_i)$ we define a function $b_{f,\chi}: Z^\lambda_{\mathbf{D}} \to \mathbf{C}$ by $b_{f,\chi}(\pi) = \chi(\pi_f)$. Let B' be the $\mathbf{C}$-algebra (with 1) of functions $Z^{\lambda}_{\mathbf{D}} \to \mathbf{C}$ generated by the functions $b_{f,\chi}$ for various f, χ as above.

LEMMA 5.3. (a) *The algebra B' is finitely generated.*

(b) *The set $Z^{\lambda}_{\mathbf{D}}$ is in bijection with the set of algebra homomorphisms $B' \to \mathbf{C}$. (To a point $\pi \in Z^{\lambda}_{\mathbf{D}}$ corresponds the evaluation homomorphism $b \mapsto b(\pi)$.)*

We prove (a). Let B'_0 be the subalgebra of B' generated by all $b_{f,\chi}$ where $f \in \mathscr{F}^i_j$ is a monomial of length $\leqslant \mathbf{c}-2$ and χ is a linear form on $\mathrm{Hom}(\mathbf{D}_j, \mathbf{D}_i)$. Then B'_0 is a finitely generated algebra since $b_{f,\chi}$ depends linearly in χ and $\dim \mathrm{Hom}(\mathbf{D}_j, \mathbf{D}_i) < \infty$. It is therefore enough to show that $B' = B'_0$.

It is enough to show that for any monomial of length l in $\mathscr{P}^i_k$ and any linear form χ on $\mathrm{Hom}(\mathbf{D}_k, \mathbf{D}_i)$, we have $b_{m,\chi} \in B'_0$. We argue by induction on l. For $l \leqslant \mathbf{c}-2$ the result is clear. Assume now that $l \geqslant \mathbf{c}-1$. From 4.11, we see that m is a $\mathbf{C}$-linear combination of elements of form

$$f^i_j \theta_j f^j_k = f^i_j \theta_{j,\lambda} f^j_k + \lambda_j f^i_j f^j_k$$

where f^i_j, f^j_k, $f^i_j f^j_k$ are monomials of length $\leqslant l-2$. Let χ be a linear form on $\mathrm{Hom}(\mathbf{D}_k, \mathbf{D}_i)$. We can find linear forms $\chi_1, \chi_2, ..., \chi_r$ on $\mathrm{Hom}(\mathbf{D}_k, \mathbf{D}_j)$ and linear forms $\chi'_1, \chi'_2, ..., \chi'_r$ on $\mathrm{Hom}(\mathbf{D}_j, \mathbf{D}_i)$ such that

$$\chi(AB) = \chi'_1(A)\,\chi_1(B) + \chi'_2(A)\,\chi_2(B) + \cdots + \chi'_r(A)\,\chi_r(B)$$

for all $A \in \mathrm{Hom}(\mathbf{D}_j, \mathbf{D}_i)$, $B \in \mathrm{Hom}(\mathbf{D}_k, \mathbf{D}_j)$. For any $\pi \in Z^{\lambda}_{\mathbf{D}}$, we have

$$\pi_{f^i_j \theta_j f^j_k} = \pi_{f^i_j \theta_{j,\lambda} f^j_k} + \lambda_j \pi_{f^i_j f^j_k} = \pi_{f^i_j} \pi_{f^j_k} + \lambda_j \pi_{f^i_j f^j_k}$$

hence

$$\begin{aligned}\chi(\pi_{f^i_j \theta_j f^j_k}) = {} & \chi'_1(\pi_{f^i_j})\,\chi_1(\pi_{f^j_k}) + \chi'_2(\pi_{f^i_j})\,\chi_2(\pi_{f^j_k}) \\ & + \cdots + \chi'_r(\pi_{f^i_j})\,\chi_r(\pi_{f^j_k}) + \lambda_j \chi(\pi_{f^i_j f^j_k}).\end{aligned}$$

In other words,

$$b_{f^i_j \theta_j f^j_k, \chi} = b_{f^i_j, \chi'_1} b_{f^j_k, \chi_1} + b_{f^i_j, \chi'_2} b_{f^j_k, \chi_2} + \cdots + b_{f^i_j, \chi'_r} b_{f^j_k, \chi_r} + \lambda_j b_{f^i_j f^j_k, \chi}.$$

By the induction hypothesis, the right hand side is in B'_0. Hence $b_{f^i_j \theta_j f^j_k, \chi} \in B_0$. Since $b_{m,\chi}$ is a $\mathbf{C}$-linear combination of such $b_{f^i_j \theta_j f^j_k, \chi}$, we have $b_{m,\chi} \in B'_0$. This proves (a).

We prove (b). Let κ: $B' \to \mathbf{C}$ be an algebra homomorphism. Let A^i_j be the coordinate ring of the affine variety $\mathrm{Hom}(\mathbf{D}_j, \mathbf{D}_i)$. For any $f \in \mathscr{F}^i_j$ we have a map $A^i_j \to B'$; it associates to ψ: $\mathrm{Hom}(\mathbf{D}_j, \mathbf{D}_i) \to \mathbf{C}$ the function $\pi \mapsto \psi(\pi_f)$ on $Z^{\lambda}_{\mathbf{D}}$. Composing this map with κ, we obtain an algebra homomorphism

$A^i_j \to \mathbf{C}$. This must be given by evaluation at a well defined element $\pi_f \in \mathrm{Hom}(\mathbf{D}_j, \mathbf{D}_i)$. Let π be the collection (π_f) where f varies. It is easy to see that $\pi \in Z^\lambda_{\mathbf{D}}$. Then κ is given by evaluation at π. Conversely, any $\pi \in Z^\lambda_{\mathbf{D}}$ defines an algebra homomorphism $B' \to \mathbf{C}$. Thus we obtain the required bijection. The lemma is proved.

5.4. From Lemma 5.3, we see that $Z^\lambda_{\mathbf{D}}$ has a natural structure of affine algebraic variety.

Now let $\mathbf{V} \in \mathscr{C}^0$. We set $G = G_{\mathbf{V}}$. Let $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}//G$ (resp. $\mathbf{M}_{\mathbf{D}, \mathbf{V}}//G$) be the geometric quotient of the affine variety $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$ (resp. $\mathbf{M}_{\mathbf{D}, \mathbf{V}}$) by the action of the reductive group G. Note that $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}//G$ is a closed subvariety of $\mathbf{M}_{\mathbf{D}, \mathbf{V}}//G$ since $\Lambda_{\mathbf{D}, \mathbf{V}, \lambda}$ is a closed subvariety of $\mathbf{M}_{\mathbf{D}, \mathbf{V}}$.

We define a map $\vartheta' \colon \Lambda_{\mathbf{D}, \mathbf{V}, \lambda} \to Z^\lambda_{\mathbf{D}}$ by $\vartheta'(x, p, q) = (\pi_{f^i_j})$ where $\pi_{f^i_j} = q_i f^i_j p_j \colon \mathbf{D}_j \to \mathbf{D}_i$. (See 2.12.) From the definitions it is clear that ϑ' is a morphism of affine varieties. This morphism is constant on the orbits of G hence it factors through a morphism $\vartheta \colon \Lambda_{\mathbf{D}, \mathbf{V}, \lambda}//G \to Z^\lambda_{\mathbf{D}}$.

In the following theorem, the statement that ϑ is injective will be established only for $\lambda = 0$; in the case where $\lambda \neq 0$, it will be established only modulo a statement (see 5.9) which I have not completely verified. The statement that ϑ is finite will be established without restriction on λ.

THEOREM 5.5. *ϑ is a finite, infective morphism. In particular, ϑ is a homeomorphism onto its image.*

For the proof of finiteness of ϑ, we shall need two lemmas.

LEMMA 5.6. *Let A be a commutative $\mathbf{C}$-algebra with 1 and let B be a commutative subalgebra with 1 of A. Let $(\xi_t)_{t \in T}$ be a family of elements which, together with B, generates A as an algebra. Let $t \mapsto k_t$ be a function from T to $\{1, 2, ...\}$. Let $c \in \mathbf{N}$, $c \geqslant 1$. We write $T' = \{t \in T \mid k_t < c\}$, $T'' = \{t \in T \mid k_t \geqslant c\}$. Assume that:*

(i) *T' is finite;*

(ii) *for any $t \in T''$, there exists $b \in B$ such that $\xi_t - b$ is a $\mathbf{C}$-linear combination of elements $\xi_{t'}$ with $t' \in T$, $k_{t'} < k_t$;*

(iii) *for any $t \in T'$, there exists $n = n(t) \geqslant 1$ such that ξ^n_t is a $\mathbf{C}$-linear combination of products $\xi^{n_1}_{t_1} \xi^{n_2}_{t_2} \cdots \xi^{n_p}_{t_p} \xi^s_t$ where $t_1, t_2, ..., t_p \in T''$, $n_1, n_2, ..., n_s \in \mathbf{N}$, $s \in [0, n-1]$ and $n_1 k_{t_1} + n_2 k_{t_2} + n_p k_{t_p} + s k_s = n$.*

Then A is a finitely generated B-module.

Let X be the set of all. functions $f \colon T \to \mathbf{N}$ which are zero for all but finitely many $t \in T$. For $f \in X$, we set $\deg(f) = \sum_{t \in T} f(t)\, k_t \in \mathbf{N}$ and we consider the product $\mu_f = \prod_{t \in T} \xi_t^{f(t)} \in A$.

By our assumption, the elements μ_f, $(f \in X)$ generate A as a B-module. Let A_0 be the B-submodule of A generated by the μ_f such that $f(t) \leqslant n(t)-1$ for all $t \in T'$ and $f(t)=0$ for all $t \in T''$. Then A_0 is finitely generated as a B-module and it is enough to show that $A_0 = A$.

It is thus enough to show that $\mu_f \in A_0$ for any $f \in X$. We show this by induction on $\deg(f)$. If $\deg(f)=0$, then $\mu_f = 1$ and the result is clear. We now assume that $\deg(f) > 0$.

Assume first that $f(t) > 0$ for some $t \in T''$. Let $b \in B$ be as in (ii). Let $f_0 \in X$ be defined by

$$f_0(\tilde{t}) = f(\tilde{t}) \qquad \text{for} \quad \tilde{t} \neq t, \qquad f_0(t) = f(t) - 1.$$

By (ii), $\mu_f - b\mu_{f_0}$ is a $\mathbf{C}$-linear combination of elements $\mu_{f_{t'}}$ (for various $t' \in T$ with $k_{t'} < k_t$) where $f_{t'} \in X$ is defined by

$$f_{t'}(\tilde{t}) = f(\tilde{t}) \qquad \text{for} \quad \tilde{t} \neq t, t', \qquad f_{t'}(t) = f(t) - 1, \qquad f_{t'}(t') = f(t') + 1.$$

We have $\deg(f_0) < \deg(f)$, $\deg(f_{t'}) < \deg(f)$ (since $k_{t'} < k_t$). By the induction hypothesis, $\mu_{f_0} \in A_0$, $\mu_{f_{t'}} \in A_0$, hence $\mu_f \in A_0$.

Assume next that $f(t) \geqslant n(t)$ for some $t \in T'$. We substitute $\xi_t^{n(t)}$ in μ_f by the expression provided by (ii). We see that μ_f is a $\mathbf{C}$-linear combination of elements of the form $\mu_{\tilde{f}}$ where $\tilde{f} \in X$ satisfies

$$\tilde{f}(t) < f(t), \qquad \tilde{f}(\tilde{t}) = f(\tilde{t}) \qquad \text{for} \quad \tilde{t} \in T' - \{t\}, \qquad \deg(\tilde{f}) = \deg(f).$$

These conditions imply that $\tilde{f}(\tilde{t}) > 0$ for some $\tilde{t} \in T''$, hence the previous argument applies to $\tilde{f}$ and gives $\mu_{\tilde{f}} \in A_0$. It follows that $\mu_f \in A_0$.

Finally, we are left with the case where $f(t) = 0$ for all $t \in T''$ and $f(t) < n(t)$ for all $t \in T'$. In this case we have by definition $\mu_f \in A_0$. This completes the induction. The lemma is proved.

Lemma 5.7. *Let V be a $\mathbf{C}$-vector space of finite dimension n_0 and let $c \in \mathbf{N}$, $c \geqslant 1$. For any integer $n' \in \mathbf{N}$, let $Y_{n'}$ be the set of all functions w: $[c, c+n_0-1] \to \mathbf{N}$ such that $\sum_{z \geqslant c} zw(z) = n'$.*

There exists an integer $n \geqslant 1$ and functions d_s: $Y_{n-s} \to \mathbf{C}$ (for $s \in [0, n-1]$) such that for any $A \in \mathrm{End}(V)$ we have

$$\mathrm{Tr}(A)^n = \sum_{s \in [0, n-1],\, w \in Y_{n-s}} d_s(w)\, \mathrm{Tr}(A)^s \prod_{z=c}^{c+n_0-1} \mathrm{Tr}(A^z)^{w(z)}.$$

The proof is left to the reader.

For example, if $n_0 = 2$ and $c = 2$ we have $\mathrm{Tr}(A)^3 = 3\,\mathrm{Tr}(A)\,\mathrm{Tr}(A^2) - 2\,\mathrm{Tr}(A^3)$. If $n_0 = 2$ and $c = 3$ we have

$$\mathrm{Tr}(A)^6 = 8\,\mathrm{Tr}(A)^3\,\mathrm{Tr}(A^3) - 9\,\mathrm{Tr}(A)^2\,\mathrm{Tr}(A^4) + 2\,\mathrm{Tr}(A^3)^2.$$

5.8. *Proof of finiteness of* ϑ. Let A, B' be the coordinate rings of $\Lambda_{\mathbf{D},\mathbf{V},\lambda}//G, Z^{\lambda}_{\mathbf{D}}$. Let B be the image of the homomorphism $B' \to A$ induced by ϑ. We must prove that A is a finitely generated B-module.

Since $\Lambda_{\mathbf{D},\mathbf{V},\lambda}//G$ is a closed subvariety of $\mathbf{M}_{\mathbf{D},\mathbf{V}}//G$, we have a natural surjective homomorphism $\mathscr{R}^G \to A$, where $\mathscr{R}^G$ is the algebra of G-invariants in the coordinate ring $\mathscr{R}$ of $\mathbf{M}_{\mathbf{D},\mathbf{V}}$. Hence A is generated as an algebra by the images of the generators of the algebra $\mathscr{R}^G$ described in 1.3. Thus, A is generated by elements $\xi_t (t \in T)$ where t is a non-empty cycle and ξ_t is the corresponding function in 1.2(a) restricted to $\Lambda_{\mathbf{D},\mathbf{V},\lambda}$ and by elements ξ'_s where s is a path and ξ'_s is the corresponding function in 1.2(b) restricted to $\Lambda_{\mathbf{D},\mathbf{V},\lambda}$. For a cycle $t \in T$ let $k_t \in \{1, 2, ...\}$ be the length of the cycle. From the definitions, it is clear that B is just the subalgebra of A generated by the various ξ'_s. Hence A is generated as a B-algebra by the elements ξ_t. Let $c = \mathbf{c} - 1 \geqslant 1$.

We show that the assumptions of Lemma 5.6 are satisfied. The assumption 5.6(i) is obviously satisfied, and the assumption 5.6(iii) follows from the definitions using Lemma 5.7. Now let $t = (h_1, h_2, ..., h_r) \in T$ be a cycle of length $r \geqslant c = \mathbf{c} - 1$. (See 1.2(a).) By 4.11, the monomial $m \in \mathscr{F}$ defined by t belongs to $\mathscr{J}$ hence is a $\mathbf{C}$-linear combination of elements

$$f^i_j \theta_j f^j_i = f^i_j \theta_{j,\lambda} f^j_i + \lambda_j f^i_j f^j_i.$$

Hence for $(x, p, q) \in \Lambda_{\mathbf{D},\mathbf{V},\lambda}$ and $v \in \mathbf{V}_{h'_1}$ we have (in the $\mathscr{F}$-module structure on $\mathbf{V}$ defined by x)

$$h_r h_{r-1} \cdots h_1(v) = \sum{}^* \left(f^i_j \theta_{j,\lambda} f^j_i(v) + \lambda_j f^i_j f^j_i(v)\right)$$

$$= \sum{}^* \left(f^i_j p_j q_j f^j_i(v) + \lambda_j f^i_j f^j_i(v)\right),$$

where $\sum^*$ denotes "$\mathbf{C}$-linear combination of." Taking traces on $\mathbf{V}_{h'_1}$, we obtain

$$\xi_t(x, p, q) = \sum{}^* \left(\mathrm{Tr}(f^i_j p_j q_j f^j_i) + \lambda_j \,\mathrm{Tr}(f^i_j f^j_i)\right)$$

$$= \sum{}^* \left(\mathrm{Tr}(q_j f^j_i f^i_j p_j) + \lambda_j \,\mathrm{Tr}(f^i_j f^j_i)\right)$$

that is, ξ_t is a $\mathbf{C}$-linear combination of elements of form $\xi'_s \in B$ and elements of form $\xi_{t'}$ with $t' \in T$ such that $k_{t'} = k_t - 2$. Thus, the assumption 5.6(ii) is satisfied in our case. We may apply Lemma 5.6 and we see that A is finitely generated as a B-module. Thus, the finiteness of ϑ is established.

5.9. For the proof of injectivity of ϑ, we shall need the following statement.

(a) *There is a unique closed G-orbit in* $\Lambda_{0,\mathbf{V},\lambda}$.

In the case where $\lambda=0$, this follows from 2.22 and 4.11: the only closed orbit is $\{0\}$.

For general λ this should follow along the following lines (but I have not verified all the details).

Let R be the set of all $\alpha\in\mathbf{Z}[I]$ that are of the form $w(i)$ for some $w\in W$ and some $i\in I$ (that is, the set of roots). Let R_λ be the set of all $\alpha=\sum_i \alpha_i i$ in R such that $\sum_i \alpha_i\lambda_i=0$.

Assume first that λ satisfies the following condition: there exists a subset I' of I such that R_λ is the intersection of R with the subgroup of $\mathbf{Z}[I]$ generated by I'. Then one should necessarily have $\mathbf{V}_i=0$ for all $i\notin I'$ and therefore $\Lambda_{0,\mathbf{V},\lambda}=\Lambda_{0,\mathbf{V},0}$ in which case the desired result is already known.

If λ does not satisfy the condition above, then we can find $w\in W$ so that $\lambda'=w(\lambda)$ does satisfy that condition. Using a sequence of "reflection functors" corresponding to a reduced expression of w, one should deduce the desired result for λ from the corresponding result for λ'.

5.10. *Proof of injectivity of* ϑ. The proofs in this subsection will be modulo the validity of 5.9(a). (They are unconditional for $\lambda=0$.)

It is enough to prove the following statement.

Let $(x,p,q),(x',p',q')$ *be two points in* $\Lambda_{\mathbf{D},\mathbf{V},\lambda}$. *Let* π,π' *be the corresponding points of* $Z^\lambda_{\mathbf{D}}$. *(See* 2.12.*) Assume that* $\pi=\pi'$. *Assume also that the the G-orbit of* (x,p,q) *and the G-orbit of* (x',p',q') *are closed in* $\Lambda_{\mathbf{D},\mathbf{V},\lambda}$. *Then* $(x,p,q),(x',p',q')$ *are in the same G-orbit.*

We associate $\mathbf{W},\mathbf{T}$ to (x,p,q) as in Lemma 2.30, and we attach in the same way $\mathbf{W}',\mathbf{T}'$ to (x',p',q'). We have

$$(x|_{\mathbf{W}},p,q|_{\mathbf{W}})\in\Lambda^s_{\mathbf{D},\mathbf{W},\lambda}\cap\Lambda^{*s}_{\mathbf{D},\mathbf{W},\lambda},\ (x'|_{\mathbf{W}'},p',q'|_{\mathbf{W}'})\in\Lambda^s_{\mathbf{D},\mathbf{W}',\lambda}\cap\Lambda^{*s}_{\mathbf{D},\mathbf{W}',\lambda}.$$

Consider the subspace $\tilde{\mathbf{W}}_i$ of $\mathbf{V}_i$ spanned by the elements $f^i_l p_l(d_l)$ with $f^i_l\in\mathscr{F}^i_l$, $d_l\in\mathscr{D}_l$ (for various l). (The action of f^i_l is given by the $\mathscr{F}$-module structure on $\mathbf{V}$ defined by x.) Then $\tilde{\mathbf{W}}=\bigoplus_{i\in I}\tilde{\mathbf{W}}_i$ is an x-adapted subspace of $\mathbf{W}$ containing $\mathrm{Im}(p)$. Since $(x|_{\mathbf{W}},p,q|_{\mathbf{W}})\in\Lambda^s_{\mathbf{D},\mathbf{W},0}$, we must have $\tilde{\mathbf{W}}=\mathbf{W}$.

We define a linear map $\gamma_i\colon\mathbf{W}_i\to\mathscr{E}''^{\mathbf{D}}_i$ by $w_i\mapsto\sum_{k\in I}\phi_k$ where $\phi_k\in\mathrm{Hom}(\mathscr{F}^k_i,\mathbf{D}_k)$ is given by $\phi_k(f^k_i)=q_k(f^k_i w_i)$ for $w_i\in\mathbf{W}_i$. Now $\mathrm{Ker}(\gamma_i)$ is an I-graded, x-adapted subspace of $\mathbf{W}$ contained in $\mathrm{Ker}(q)$. Since $(x|_{\mathbf{W}},p,q|_{\mathbf{W}})\in\Lambda^{*s}_{\mathbf{D},\mathbf{W},0}$, we must have $\mathrm{Ker}(\gamma_i)=0$. It follows that $\dim\mathbf{W}_i=\dim\gamma_i(\mathbf{W}_i)=\dim\gamma_i(\tilde{\mathbf{W}}_i)$. Now $\gamma_i(\tilde{\mathbf{W}}_i)$ is the subspace of $\mathscr{E}''^{\mathbf{D}}_i$ spanned by the elements $\sum_{k\in I}\phi_k$ where $\phi_k\in\mathrm{Hom}(\mathscr{F}^k_i,\mathbf{D}_k)$ is given by

$$\phi_k(f^k_i)=q_k(f^k_i f^i_l p_l(d_l))=\pi_{f^k_i f^i_l}(d_l)$$

with $f_i^k \in \mathscr{F}_i^k$, $f_l^i \in \mathscr{F}_l^i$, $d_l \in \mathbf{D}_l$ (for various l). Equivalently, $\gamma_i(\tilde{\mathbf{W}}_i)$ is the subspace of $\mathscr{E}_i''^{\mathbf{D}}$ spanned by the elements $\alpha_\pi(f_l^i \otimes d_l)$ for various $f_l^i \in \mathscr{F}_l^i$, $d_l \in \mathbf{D}_l$. Thus, $\gamma_i(\tilde{\mathbf{W}}_i) = \alpha_\pi(\mathscr{E}_i^{\mathbf{D}})$.

We see therefore that $\dim \mathbf{W}_i = \dim \alpha_\pi(\mathscr{E}_i^{\mathbf{D}})$. Similarly, we have $\dim \mathbf{W}'_i = \dim \alpha_{\pi'}(\mathscr{E}_i^{\mathbf{D}})$. Since $\pi = \pi'$, we have $\alpha_\pi = \alpha_{\pi'}$, hence $\dim \mathbf{W}_i = \dim \mathbf{W}'_i$. Since this holds for any i, we have $|\mathbf{W}| = |\mathbf{W}'|$ and hence $|\mathbf{T}| = |\mathbf{T}'|$.

Now the element in $Z_{\mathbf{D}}^\lambda$ attached to $(x|_{\mathbf{W}}, p, q|_{\mathbf{W}})$ as in 2.12 is clearly just π above. Similarly, the element in $Z_{\mathbf{D}}^\lambda$ attached to $(x'|_{\mathbf{W}}, p', q'|_{\mathbf{W}})$ as in 2.12 is just π'. (Recall that $\pi = \pi'$.)

By Lemma 2.28, we can find an isomorphism $\mathbf{W} \xrightarrow{\sim} \mathbf{W}'$ in $\mathscr{C}^0$ which carries $(x|_{\mathbf{W}}, p, q|_{\mathbf{W}})$ to $(x'|_{\mathbf{W}'}, p', q'|_{\mathbf{W}'})$.

By 5.9(a), we can find an isomorphism $\mathbf{T} \xrightarrow{\sim} \mathbf{T}'$ in $\mathscr{C}^0$ which carries $(x|_{\mathbf{T}}, 0, 0)$ to $(x'|_{\mathbf{T}'}, 0, 0)$. Taking the direct sum of these two isomorphisms, we find an isomorphism $\mathbf{V} \xrightarrow{\sim} \mathbf{V}$ in $\mathscr{C}^0$ which carries (x, p, q) to (x', p', q').

Hence (x, p, q), (x', p', q') are in the same G-orbit. The injectivity of ϑ is established. Theorem 5.5 is proved.

5.11. Let X_i be the subspace of $\mathscr{E}^{\mathbf{D}}$ spanned by the elements $f_k^i \otimes d_k$ with $f_k^i \in \mathscr{P}_k^i$ of length $\leqslant \mathbf{c} - 2$ and $d_k \in \mathbf{D}_k$ (k variable). Let $X = \bigoplus_{i \in I} X_i$. Note that $X \in \mathscr{C}^0$.

PROPOSITION 5.12. *For any $\pi \in Z_{\mathbf{D}}^\lambda$ we have $X + \mathscr{I}^\pi = \mathscr{E}^{\mathbf{D}}$.*

Let $f_k^i \in \mathscr{P}_k^i$ be of length l and let $d_k \in \mathbf{D}_k$. It is enough to show that $f_k^i \otimes d_k \in X_i + \mathscr{I}_i^\pi$. We argue by induction on $l \geqslant 0$. If $l \leqslant \mathbf{c} - 2$, the result is obvious. Assume now that $l \geqslant \mathbf{c} - 1$. By 4.11, $f_k^i \otimes d_k$ is a $\mathbf{C}$-linear combination of elements

$$f_j^i \theta_j f_k^j \otimes d_k = f_j^i \theta_{j,\lambda} f_k^j \otimes d_k + \lambda_j f_j^i f_k^j \otimes d_k,$$

where f_j^i, f_k^j, $f_j^i f_k^j$ are monomials of length $< l$. By the induction hypothesis, we have $f_j^i f_k^j \otimes d_k \in X + \mathscr{I}^\pi$, so it remains to show that $f_j^i \theta_{j,\lambda} f_k^j \otimes d_k \in X + \mathscr{I}^\pi$. We have

$$f_j^i \theta_{j,\lambda} f_k^j \otimes d_k = -\beta_\pi(f_j^i \otimes f_k^j \otimes d_k) - f_j^i \otimes \pi_{f_k^j}(d_k).$$

We have $\beta_\pi(f_j^i \otimes f_k^j \otimes d_k) \in \mathscr{I}^\pi$ and $f_j^i \otimes \pi_{f_k^j}(d_k) \in X + \mathscr{I}^\pi$ by the induction hypothesis. The proposition is proved.

COROLLARY 5.13. *The union of the images of the maps $\vartheta: \Lambda_{\mathbf{D},\mathbf{V},\lambda}//G \to Z_{\mathbf{D}}^\lambda$ (for fixed λ, $\mathbf{D}$ but variable $\mathbf{V}$) is the whole of $Z_{\mathbf{D}}^\lambda$.*

Let $\pi \in Z_{\mathbf{D}}^\lambda$. By, 5.12, we have $\mathscr{E}^{\mathbf{D}}/\mathscr{I}^\pi \in \mathscr{C}^0$. Let $\mathscr{V} \subset \mathscr{E}^{\mathbf{D}}$ be given by $\mathscr{V} = \mathscr{I}^\pi$. Let $(x, p, q) \in \Lambda_{\mathbf{D},\mathscr{E}^{\mathbf{D}}/\mathscr{V},\lambda}$ be defined as in 2.18. As pointed out in 2.18, the element of $Z^{\lambda}_{\mathbf{D}}$ attached to this (x, p, q) is just π. The corollary follows.

COROLLARY 5.14. *Let* $\mathbf{V} \in \mathscr{C}^0$. *If* $\Lambda^s_{\mathbf{V}, \mathbf{D}, \lambda} \neq \emptyset$, *then* $\dim \mathbf{V} \leqslant \dim X$.

Indeed, let $(\pi, \mathscr{V})$ correspond to $(\mathbf{V}, x, p, q)$ under 2.20. Since $\mathscr{I}^{\pi} \subset \mathscr{V}$ and $\mathbf{V}$ is isomorphic to $\mathscr{E}^{\mathbf{D}}/\mathscr{V}$, we have $\dim \mathbf{V} = \dim \mathscr{E}^{\mathbf{D}}/\mathscr{V} \leqslant \dim \mathscr{E}^{\mathbf{D}}/\mathscr{I}^{\pi} \leqslant \dim X$. The last inequality follows from 5.12. The corollary is proved.

6. THE AFFINE CASE

6.1. Let Γ be a finite group. Let $\mathscr{C}_{\Gamma}$ be the category whose objects are $\mathbf{C}$-vector spaces with a given linear action of Γ and whose morphisms are $\mathbf{C}$-linear maps compatible with the Γ-action. Let $\mathscr{C}^0_{\Gamma}$ be the full subcategory of $\mathscr{C}_{\Gamma}$ whose objects are finite dimensional $\mathbf{C}$-vector spaces with a given linear action of Γ. For $M, M' \in \mathscr{C}_{\Gamma}$, we write $\mathrm{Hom}_{\Gamma}(M, M')$ instead of $\mathrm{Hom}_{\mathscr{C}_{\Gamma}}(M, M')$.

Let I be the set of isomorphism classes of irreducible Γ-modules over $\mathbf{C}$. For each $i \in I$ we assume given a simple Γ-module ρ_i in the class i. Let $D_i = \dim \rho_i$.

We have equivalences of categories $\mathscr{A}: \mathscr{C} \xrightarrow{\sim} \mathscr{C}_{\Gamma}$, $\mathscr{A}': \mathscr{C}_{\Gamma} \xrightarrow{\sim} \mathscr{C}$, inverse to each other,

$$\mathbf{V} \mapsto \mathscr{A}(\mathbf{V}) = \bigoplus_{i \in I} \mathrm{Hom}(\rho_i, \mathbf{V}_i),$$

$$M \to \mathscr{A}'(M) = \bigoplus_{i \in I} \mathrm{Hom}_{\Gamma}(\mathbf{C}, \rho_i \otimes M),$$

where $\mathbf{C}$ is considered with the trivial Γ-action.

Let A be a ring in $\mathscr{C}_{\Gamma}$, that is, an object of $\mathscr{C}_{\Gamma}$, with a given morphism $A \otimes A \to A$ in $\mathscr{C}_{\Gamma}$ which make A into an associative $\mathbf{C}$-algebra.

Let $\tilde{A} = \bigoplus_{i, j \in I} \tilde{A}^i_j$ where $\tilde{A}^i_j = \mathrm{Hom}_{\Gamma}(\rho_j, \rho_i \otimes A)$. We regard $\tilde{A}$ as a $\mathbf{C}$-algebra in which the product of $f \in \mathrm{Hom}_{\Gamma}(\rho_j, \rho_i \otimes A)$ and $f' \in \mathrm{Hom}_{\Gamma}(\rho_k, \rho_{j'} \otimes A)$ is the composition

$$\rho_k \xrightarrow{f'} \rho_j \otimes A \xrightarrow{f \otimes 1_A} \rho_i \otimes A \otimes A \to \rho_i \otimes A, \qquad \text{if} \quad j = j',$$

(the last map is given by multiplication in A), and is zero if $j \neq j'$.

We say that $M \in \mathscr{C}_{\Gamma}$ is an A-module in $\mathscr{C}_{\Gamma}$ if we are given a morphism $A \otimes M \to M$ in $\mathscr{C}_{\Gamma}$ which makes the $\mathbf{C}$-vector space M into an A-module in the usual sense. If M is an A-module in $\mathscr{C}_{\Gamma}$, then $\mathscr{A}'(M)$ is an $\tilde{A}$-module in which the product of $f \in \mathrm{Hom}_\Gamma(\rho_j, \rho_i \otimes A)$ and $g \in \mathrm{Hom}_\Gamma(\mathbf{C}, \rho_{j'} \otimes M)$ is the composition

$$\mathbf{C} \xrightarrow{g} \rho_j \otimes M \xrightarrow{f \otimes 1_M} \rho_i \otimes A \otimes M \to \rho_i \otimes M, \qquad \text{if} \quad j = j'$$

(the last map is given by the A-module structure of M) and is zero, if $j \neq j'$. For such M, the following statement is easily verified:

(a) *The map $M_1 \mapsto \mathscr{A}'(M_1)$ establishes a bijection between the set of Γ-stable subspaces of M that are A-submodules and the set of $\mathbf{I}$-graded subspaces of $\mathscr{A}'(M)$ that are $\tilde{A}$-submodules.*

6.2. From now on, we assume that Γ is a finite subgroup of the symplectic group $Sp(T)$ where T is a two-dimensional $\mathbf{C}$-vector space with a given non-singular symplectic form $\langle\ ,\ \rangle\colon T \times T \to \mathbf{C}$. For $u \in \mathbf{N}$ we set $T^u = T \otimes T \otimes \cdots \otimes T$ (u factors) and let S^u be the u-th symmetric power of T, regarded as a quotient of T^u. Let $S^\dagger = \bigoplus_{u \in \mathbf{N}} S^u$. This is naturally a quotient algebra of the tensor algebra $T^\dagger = \bigoplus_{u \in \mathbf{N}} T^u$. Note that S^u, T^u are naturally Γ-modules since T is a Γ-module. Moreover, $S^\dagger$, $T^\dagger$ are rings in $\mathscr{C}_\Gamma$. If $M \in \mathscr{C}^0_\Gamma$ then $S^\dagger \otimes M$ is naturally an $S^\dagger$-module in $\mathscr{C}_\Gamma$ (Γ acts on both factors).

Definition 6.3. Let $M, M' \in \mathscr{C}^0_\Gamma$. Let $\mathscr{H}^{M'}(M)$ be the set of all $S^\dagger$-submodules $\mathscr{W}$ of $S^\dagger \otimes M$ which are also Γ-submodules such that $(S^\dagger \otimes M)/\mathscr{W}$ is isomorphic to M' as a Γ-module. Let $\mathscr{H}^{M'}_0(M)$ be the set of all $\mathscr{W} \in \mathscr{H}^{M'}(M)$ such that $\mathscr{W}$ contains $S^u \otimes M$ for all large enough u.

Note that in the last definition u can be taken independent of $\mathscr{W}$; it follows that $\mathscr{H}^{M'}_0(M)$ is a projective variety (a closed subvariety of a grassmannian).

When $\Gamma = \{1\}$ and $M = \mathbf{C}$, $M' = \mathbf{C}^n$, then $\mathscr{H}^{M'}(M)$ is just the Hilbert scheme of n points on T and $\mathscr{H}^{M'}_0(M)$ is just the fibre at 0 of the canonical map of the Hilbert scheme to the n-th symmetric power of T. Thus, $\mathscr{H}^{M'}(M)$ is a generalization of the Hilbert scheme.

The main result of this section is that, if $\Gamma \neq \{1\}$, then $\mathscr{H}^{M'}_0(M)$ may be canonically identified with the moduli space (or set of similarity classes) of "nilpotent" stable quadruples $\mathscr{R}'^{n}_{\mathbf{D}, v}$ for a graph (I, H) determined by Γ as in McKay's correspondence and for $\mathbf{D}$, v determined by M, M'.

The analogy between quiver varieties and Hilbert schemes has already been pointed out by Nakajima [N3].

6.4. We shall use the following notation: if V, V' are finite dimensional $\mathbf{C}$-vector spaces with a given non-degenerate bilinear pairing $(\ ,\ )\colon V \times V' \to \mathbf{C}$, we define $\varpi^{(\ ,\ )} \in V \times V'$ by $\varpi^{(\ ,\ )} = \sum_{s \in \Sigma} e^s \otimes e'^s$ where $(e^s)_{s \in \Sigma}$, $(e'^s)_{s \in \Sigma}$ are bases of V, V' such that $(e^s, e'^{s'}) = \delta_{s, s'}$ for all s, s'. (Then $\varpi^{(\ ,\ )}$ is independent of the choice of bases.)

In the following lemma, the symplectic vector space T does not play a role.

LEMMA 6.5. *Let $M \in \mathscr{C}_\Gamma$. Let $D \in \mathbf{C}^*$. We define a bilinear form $(\ ,\ )'_j$: $\mathrm{Hom}_\Gamma(\rho_j, M) \times \mathrm{Hom}_\Gamma(M, \rho_j) \to \mathbf{C}$ by $(f, f')'_j = (D_j D)^{-1}\,\mathrm{Tr}(ff'\colon M \to M)$. Let a'_j: $\mathrm{Hom}(\rho_j, M) \otimes \mathrm{Hom}_\Gamma(M, \rho_j) \to \mathrm{Hom}_\Gamma(M, M)$ be the linear map which attaches to $f \otimes f'$ the composition $ff'\colon M \to M$. Then*:

(a) *The bilinear form $(\ ,\ )'_j$ is non-singular.*

(b) *Let $\varpi^{(\ ,\ )'_j} \in \mathrm{Hom}_\Gamma(\rho_j, M) \otimes \mathrm{Hom}_\Gamma(M, \rho_j)$ be defined as in* 6.4. *Then $\sum_{j \in I} a'_j(\varpi^{(\ ,\ )'_j}) \in \mathrm{Hom}_\Gamma(M, M)$ is equal to $D1_M$.*

(a) is immediate. We prove (b). It is easy to see that, if the lemma is true for some $D \in \mathbf{C}^*$, then it is true for any $D \in \mathbf{C}^*$. Hence we may assume that $D = 1$. We can write canonically $M = \bigoplus_{j \in I} M^j$ (in $\mathscr{C}_\Gamma$) where M^j is the ρ_j-isotypic component of M. We have canonically

$$\mathrm{Hom}_\Gamma(\rho_j, M^j) = \mathrm{Hom}_\Gamma(\rho_j, M), \qquad \mathrm{Hom}_\Gamma(M^j, \rho_j) = \mathrm{Hom}_\Gamma(M, \rho_j).$$

Then the form $(\ ,\ )'_j$ defined in terms of M^j becomes identified with the form $(\ ,\ )'_j$ defined in terms of M and the element $\varpi^{(\ ,\ )'_j}$ defined in terms of M^j becomes identified with the element $\varpi^{(\ ,\ )'_j}$ defined in terms of M. If the result is true for each M^j instead of M, we see that (with the notation of (b)), $a'_j(\varpi^{(\ ,\ )'_j}) \in \mathrm{Hom}_\Gamma(M, M)$ is equal to to the composition of the canonical projection $M \to M^j$ with the canonical imbedding $M^j \to M$. Summing these over all j we get the identity map of M. Hence (b) would hold for M. Thus we are reduced to the case where M is ρ_j-isotypic. In this case the proof is immediate. The lemma is proved.

6.6. Let $t = \dim T$. In 6.6–6.9, we do not need the assumption that $t = 2$.

We write $\rho_i T^u$ instead of $\rho_i \otimes T^u$. (An object of $\mathscr{C}^0_\Gamma$.) For $i, j \in I$ we set $X_i^j = \mathrm{Hom}_\Gamma(\rho_i, \rho_j T)$, $X'^j_i = \mathrm{Hom}_\Gamma(\rho_i T, \rho_j)$.

We define a bilinear form $(\ ,\ )_j$: $X_j^i \times X_i^j \to \mathbf{C}$ or, equivalently, a linear function $X_j^i \otimes X_i^j \to \mathbf{C}$ as the composition

$$X_j^i \otimes X_i^j \xrightarrow{a_j} \mathrm{Hom}_\Gamma(\rho_i, \rho_i T^2) \xrightarrow{b} \mathrm{Hom}_\Gamma(\rho_i, \rho_i) \xrightarrow{c} \mathbf{C},$$

where a_j, b, c are the linear maps defined as follows.

a_j attaches to $f \otimes f' \in X_j^i \otimes X_i^j$ the composition $\rho_i \xrightarrow{f'} \rho_j T \xrightarrow{f \otimes 1_T} \rho_i T^2$.

b attaches to $f'' \in \mathrm{Hom}_\Gamma(\rho_i, \rho_i T^2)$ the composition $\rho_i \xrightarrow{f''} \rho_i T^2 \xrightarrow{1 \otimes \langle\ ,\ \rangle} \rho_i$.

c is $-(tD_i D_j)^{-1}$ times the trace of a linear transformation of the $\mathbf{C}$-vector space ρ_i.

PROPOSITION 6.7. (a) *For* $f \in X^i_j$, $f' \in X^j_i$ *we have* $(f, f')_j = -(f', f)_i$.

(b) *The bilinear form* $(\ ,\ )_j\colon X^i_j \times X^j_i \to \mathbf{C}$ *is non-singular.*

(c) *Let* $\varpi^{(\ ,\ )_j} \in X^i_j \otimes X^j_i$, $\varpi^{\langle\ ,\ \rangle} \in T \otimes T$ *be defined as in* 6.4. *Let* $\theta_i = \sum_{j \in I} a_j(\varpi^{(\ ,\ )_j}) \in \mathrm{Hom}_\Gamma(\rho_i, \rho_i T^2)$. *Then* θ_i *is the map* $x \mapsto -tD_i x \otimes \varpi^{\langle\ ,\ \rangle}$.

We prove (a). Let $(e_s)_{s \in \Sigma}$ be a basis of T. We can write

$$f(x) = \sum_{s \in \Sigma} f_s(x) \otimes e_s, \qquad f'(x') = \sum_{s \in \Sigma} f'_s(x') \otimes e_s$$

for $x \in \rho_j$, $x' \in \rho_i$, where $f_s\colon \rho_j \to \rho_i$, $f'_s\colon \rho_i \to \rho_j$ are $\mathbf{C}$-linear maps.
From the definitions we have

$$(f, f')_j = -(tD_i D_j)^{-1} \sum_{s, s' \in \Sigma} \langle e_{s'}, e_s \rangle \operatorname{Tr}(f_{s'} f'_s\colon \rho_i \to \rho_i),$$

$$(f', f)_i = -(tD_i D_j)^{-1} \sum_{s, s' \in \Sigma} \langle e_{s'}, e_s \rangle \operatorname{Tr}(f'_{s'} f_s\colon \rho_j \to \rho_j).$$

We now use the equalities $\operatorname{Tr}(f'_{s'} f_s\colon \rho_j \to \rho_j) = \operatorname{Tr}(f_s f'_{s'}\colon \rho_i \to \rho_i)$ and $\langle e_{s'}, e_s \rangle = -\langle e_s, e_{s'} \rangle$ and we get (a).

From the definitions we see that we have a commutative diagram

$$\begin{tikzcd}
X^i_j \otimes X^j_i \arrow[r, "a_j"] \arrow[d, "1 \otimes d"'] & \mathrm{Hom}_\Gamma(\rho_i, \rho_i T^2) \arrow[r, "b"] \arrow[d, "\tilde{d}"'] & \mathrm{Hom}_\Gamma(\rho_i, \rho_i) \arrow[d, "c"'] \\
X^i_j \otimes X'^j_i \arrow[r, "a'_j"] & \mathrm{Hom}_\Gamma(\rho_i T, \rho_i T) \arrow[r, "c'"] & \mathbf{C}
\end{tikzcd}$$

Here a_j, b, c are as in 6.6 and the linear maps $a'_j, c', d, \tilde{d}$ are defined as follows.

a'_j attaches to $f \otimes f' \in X^i_j \otimes X'^j_i$ the composition $\rho_i T \xrightarrow{f'} \rho_j \xrightarrow{f} \rho_i T$.

c' is $(tD_i D_j)^{-1}$ times the trace of a linear transformation of the $\mathbf{C}$-vector space $\rho_i T$.

$d\colon X^j_i \to X'^j_i$ attaches to $f' \in X^j_i$ the composition $\rho_i T \xrightarrow{f' \otimes 1_T} \rho_j T^2 \xrightarrow{1 \otimes \langle\ ,\ \rangle} \rho_j$.

$\tilde{d}$ attaches to $f'' \in \mathrm{Hom}_\Gamma(\rho_i, \rho_i T^2)$ the composition $\rho_i T \xrightarrow{f'' \otimes 1_T} \rho_i T^3 \xrightarrow{1 \otimes \langle\ ,\ \rangle} \rho_i T$.

Clearly, $d, \tilde{d}$ are isomorphisms. Moreover, $\tilde{d}$ carries the map $x \mapsto -tD_i x \otimes \varpi^{\langle\ ,\ \rangle}$ (in $\mathrm{Hom}_\Gamma(\rho_i, \rho_i T^2)$) to $tD_i 1_{\rho_i T}$.

Using the commutative diagram above, we see that (b),(c) are consequences of Lemma 6.5(a),(b), with $M = \rho_i T$ and $D = tD_i$.

6.8. Since $T^\dagger$, $S^\dagger$ are rings in $\mathscr{C}_\Gamma$, we can associate to them algebras

$$\tilde{T}^\dagger = \bigoplus_{i,\,j\in I} \mathrm{Hom}_\Gamma(\rho_j, \rho_i \otimes T^\dagger), \qquad \tilde{S}^\dagger = \bigoplus_{i,\,j\in I} \mathrm{Hom}_\Gamma(\rho_j, \rho_i \otimes S^\dagger)$$

as in 6.1. The canonical surjection $T^\dagger \to S^\dagger$ induces a surjective algebra homomorphism $\tilde{T}^\dagger \to \tilde{S}^\dagger$ whose kernel is

(a) $\bigoplus_{i,\,j} \mathrm{Hom}_\Gamma(\rho_j, \rho_i \otimes K^\dagger) \subset \tilde{T}^\dagger$

where $K^\dagger = \bigoplus_{u\in\mathbf{N}} K^u$ and $K^u = \mathrm{Ker}(T^u \to S^u)$.

LEMMA 6.9. *The two sided ideal 6.8(a) is generated by* $\bigoplus_{i,\,j\in I} \mathrm{Hom}_\Gamma(\rho_j, \rho_i \otimes K^2)$.

Multiplication in $\tilde{T}^\dagger$ defines an isomorphism

$$\mathrm{Hom}_\Gamma(\rho_k, \rho_i \otimes T^{u'}) \otimes \mathrm{Hom}_\Gamma(\rho_l, \rho_k \otimes K^2)$$

$$\otimes \mathrm{Hom}_\Gamma(\rho_j, \rho_l \otimes T^{u''}) \xrightarrow{\sim} \mathrm{Hom}_\Gamma(\rho_j, \rho_i \otimes T^{u'} \otimes K^2 \otimes T^{u''}).$$

Since $K^u = \sum_{u',\,u''\in\mathbf{N};\,u'+u''+2=u} T^{u'}K^2T^{u''}$, the lemma follows.

6.10. From now on we will make use of the assumption that $t = \dim(T) = 2$. Then K^2 is one-dimensional. It is clearly spanned by the Γ-invariant element $\varpi^{\langle\,,\,\rangle} \in T^2$. Hence $\mathrm{Hom}_\Gamma(\rho_j, \rho_i \otimes K^2)$ is 0 for $i \neq j$, while for $i = j$ it is the subspace of $\mathrm{Hom}_\Gamma(\rho_i, \rho_i \otimes T^2)$ spanned by θ_i (see 6.7). Hence from Lemma 6.9 we deduce:

PROPOSITION 6.11. $\tilde{S}^\dagger$ *is the quotient of the algebra* $\tilde{T}^\dagger$ *by the two-sided ideal generated by the elements* θ_i *with* $i \in I$.

6.12. From now on we assume that $\Gamma \neq \{1\}$. Then

(a) $X_i^i = 0$ for all $i \in I$.

Indeed, if Γ contains the centre of $Sp(T)$, then that centre acts by different characters on ρ_i, $\rho_i \otimes T$ hence these two representations of Γ are disjoint. If, on the other hand, Γ does not contain the centre of Γ, then Γ is a cyclic subgroup of $Sp(T)$ of odd order $\geqslant 3$. Then ρ_i is one dimensional. If we had $X_i^i \neq 0$, then T would contain the unit representation of Γ, which is manifestly not the case. Thus, (a) is verified.

For any $i \neq j$ in I we choose a basis H_i^j of X_i^j. Let $H = \bigsqcup_{i,\,j} H_i^j$. Using Lemma 6.7(a),(b), we may assume that there exists an involution $h \mapsto \bar{h}$ of H such that the following hold:

if $h \in H_j^i$, then $\bar{h} \in H_i^j$;
$(h, \bar{h})_j = \varepsilon(h)^{-1}$ where $\varepsilon(h) \in \mathbf{C}^*$ for $h \in H_j^i$;
$(h, \tilde{h})_j = 0$ if $h \in H_j^i$, $\tilde{h} \in H_i^j$, $\tilde{h} \neq \bar{h}$.

For $h \in H_i^j$ we set $h' = i$, $h'' = j$. Then (I, H), $h \mapsto \bar{h}$, $h \mapsto h'$, $h \mapsto h''$, $h \mapsto \varepsilon(h)$ are as in 1.1, 2.1. Hence the algebras $\mathscr{F}$, $\mathbf{P}$ are defined in terms of these data as in 2.2, 2.25. Note that, for $i, j \in I$, $u \in \mathbf{N}$, multiplication in $\tilde{T}^\dagger$ defines an isomorphism

$$\bigoplus_{i_1, i_2, \ldots, i_{u-1}} X_{i_1}^i \otimes X_{i_2}^{i_1} \otimes X_{i_3}^{i_2} \otimes \cdots \otimes X_j^{i_{u-1}} \xrightarrow{\sim} \mathrm{Hom}_\Gamma(\rho_j, \rho_i T^u).$$

This gives a $\mathbf{C}$-basis given by monomials in the h, and gives an algebra isomorphism

$$\mathscr{F} \xrightarrow{\sim} \tilde{T}^\dagger. \tag{b}$$

Clearly the element $\theta_i \in \tilde{T}^\dagger$ given by 6.7(c) corresponds under this isomorphism to the element $\theta_i \in \mathscr{F}$ (see 2.4). Hence there is an induced algebra isomorphism

$$\mathbf{P} \xrightarrow{\sim} \tilde{S}^\dagger. \tag{c}$$

6.13. Let $\mathbf{D} \in \mathscr{C}^0$ and let $M = \mathscr{A}(\mathbf{D}) \in \mathscr{C}_\Gamma^0$. Using the definition (see 2.25) and 6.12(c) we have

$$\begin{aligned} \bar{\mathscr{E}}_i^{\mathbf{D}} &= \bigoplus_{k \in I} \mathrm{Hom}_\Gamma(\rho_k, \rho_i \otimes S^\dagger) \otimes \mathrm{Hom}_\Gamma(\mathbf{C}, \rho_k \otimes M) \\ &= \mathrm{Hom}_\Gamma(\mathbf{C}, \rho_i \otimes S^\dagger \otimes M). \end{aligned}$$

The last isomorphism is obtained by attaching to

$$f \otimes f' \in \mathrm{Hom}_\Gamma(\rho_k, \rho_i \otimes S^\dagger) \otimes \mathrm{Hom}_\Gamma(\mathbf{C}, \rho_k \otimes M)$$

the composition $\mathbf{C} \xrightarrow{f'} \rho_k \otimes M \xrightarrow{f \otimes 1_M} \rho_i \otimes S^\dagger \otimes M$. In other words,

$$\bar{\mathscr{E}}^{\mathbf{D}} = \mathscr{A}'(S^\dagger \otimes M). \tag{a}$$

Now $S^\dagger \otimes M$ is naturally a module over $S^\dagger$ in $\mathscr{C}^\Gamma$. This induces on $\mathscr{A}'(S^\dagger \otimes M)$ a structure of $\tilde{S}^\dagger$-module (as explained in 6.1). This corresponds to the $\mathbf{P}$-module structure on $\bar{\mathscr{E}}^{\mathbf{D}}$ via (a) and 6.12(c). Using 6.1(a) we see therefore that

(b) *the map $M_1 \mapsto \mathscr{A}'(M_1)$ establishes a bijection between the set of Γ-stable subspaces of $S^\dagger \otimes M$ that are $S^\dagger$-submodules and the set of I-graded subspaces of $\bar{\mathscr{E}}^{\mathbf{D}}$ that are $\mathbf{P}$-submodules.*

Let us also fix $\nu \in \mathbf{N}[I]$ or equivalently, an isomorphism class of an object $M' \in \mathscr{C}^0_\Gamma$. (The correspondence is $\nu = |\mathscr{A}'(M')|$.) Combining the bijection (b) with the bijection in 2.26 between the sets 2.26 (a),(b) we obtain the following result.

Proposition 6.14. *There is a canonical bijection between the set of (similarity classes of) stable quadruples in $\mathscr{R}'_{\mathbf{D}, \nu, 0}$ such that the associated $\pi \in Z^0_{\mathbf{D}}$ is zero, and the set $\mathscr{H}^{M'}(M)$ (see* 6.3).

Similarly, combining the bijection in 2.26 with 6.13(b), we obtain the following result.

Corollary 6.15. *The bijection in* 6.14 *restricts to a bijection between the set of (similarity classes of) stable quadruples in $\mathscr{R}'^{n}_{\mathbf{D}, \nu}$ and the set $\mathscr{H}^{M'}_0(M)$.*

6.16. We consider the linear $\mathbf{C}^*$-action $t: v \mapsto t \circ v$ on $S^\dagger \otimes M$ given by $t \circ v = t^u v$ for $t \in \mathbf{C}^*$, $v \in S^u \otimes M$. This action commutes with the Γ-action. For any $f \in S^l$ and any $v \in S^\dagger M$ we have $t \circ (fv) = t^l f(t \circ v)$. It follows that, if $\mathscr{W}$ is a $S^\dagger$-submodule and a Γ-submodule of $S^\dagger M$, then $t \circ \mathscr{W}$ is again a $S^\dagger$-submodule and a Γ-submodule of $S^\dagger M$. Moreover, if $\mathscr{W}$ contains $S^u \otimes M$ then so does $t \circ \mathscr{W}$. We thus obtain an (algebraic) action $t: \mathscr{W} \mapsto t \circ \mathscr{W}$ of $\mathbf{C}^*$ on $\mathscr{H}^{M'}_0(M)$.

This action seems to be different from the one defined by Nakajima in [N1, N2] since Nakajima's action depends on a choice of orientation, while ours does not.

Let $\mathscr{H}^{M'}_0(M)^{\mathbf{C}^*}$ be the fixed point of this $\mathbf{C}^*$-action.

It is clear that $\mathscr{H}^{M'}_0(M)^{\mathbf{C}^*}$ is the projective variety consisting of all $S^\dagger$-submodules $\mathscr{W}$ of $S^\dagger \otimes M$ which are also Γ-submodules, such that $\mathscr{W} = \bigoplus_{u \in \mathbf{N}} \mathscr{W}^u$ where $\mathscr{W}^u$ is a subspace of $S^u \otimes M$ for any u, equal to $S^u \otimes M$ for large enough u and such that $(S^\dagger \otimes M)/\mathscr{W}$ is isomorphic to M' as a Γ-module.

7. GRAPHS OF FINITE TYPE

7.1. The description of the sets $\mathscr{R}'^{n}_{\mathbf{D}, \nu}$ for a connected graph of finite type can be reduced to that in the affine case, as follows.

Let $\Gamma \subset Sp(T)$ be as in 6.2. Assume that $\Gamma \neq \{1\}$. Let $\mathscr{C}^0_{*\Gamma}$ be the full subcategory of $\mathscr{C}^0_\Gamma$ whose objects are those $M \in \mathscr{C}^0_\Gamma$ such that the unit representation of Γ does not appear in M.

Let $i_0 \in I$ (see 6.1) be the class of the unit representation of Γ and let $I_* = I - \{i_0\}$. Let H_* be the subset of H (see 6.12) consisting of those $h \in H$

such that $h' \neq i_0$, $h'' \neq i_0$. Then (I_*, H_*) is a graph as in 1.1, 2.1; moreover, it is connected, of finite type.

Let $\mathbf{D}$ be a finite dimensional I_*-graded $\mathbf{C}$-vector space and let $v \in \mathbf{N}[I_*]$. It is clear that $\mathscr{R}'^n_{\mathbf{D}, v}$ defined in terms of (I_*, H_*) may be canonically identified with the set $\mathscr{R}'^n_{\mathbf{D}, v}$ defined in terms of (I, H), where $\mathbf{D}$ is regarded as an I-graded vector space with $\mathbf{D}_{i_0} = 0$ and v is regarded as an element of $\mathbf{N}[I]$ in the obvious way. Let $M \in \mathscr{C}^0_\Gamma$ be defined by $M = \mathscr{A}(\mathbf{D})$ and let $M' \in \mathscr{C}^0_\Gamma$ be defined (up to isomorphism) in terms of v as in 6.13 (so that, in fact, $M, M' \in \mathscr{C}^0_{*\Gamma}$). Then, by 6.15,

$$\mathscr{R}'^n_{\mathbf{D}, v} \text{ is in canonical bijection with } \mathscr{H}_0^{M'}(M).$$

Note that in our case we have

(a) $\quad \mathscr{H}^{M'}(M) = \mathscr{H}_0^{M'}(M).$

An equivalent statement is: if $(\mathbf{V}, x, p, q)$ is a stable quadruple for (I_*, H_*), $\mathbf{D}$ and $\lambda = 0$ such that the associated element $\pi \in Z^0_{\mathbf{D}}$ is 0, then x is nilpotent and $q = 0$. This follows from 2.23(a) and 4.11.

Definition 7.2. Let $M \in \mathscr{C}^0_{*\Gamma}$. Let $\mathscr{H}(M)$ be the set of all $S^\dagger$-submodules $\mathscr{W}$ of $S^\dagger \otimes M$ which are also Γ-submodules such that $(S^\dagger \otimes M)/\mathscr{W} \in \mathscr{C}^0_{*\Gamma}$.

For $\mathscr{W} \in \mathscr{H}(M)$ we have automatically $S^u \otimes M \subset \mathscr{W}$ for all large enough u. (See 7.1(a).) Hence.

$$\mathscr{H}(M) = \bigsqcup_{M'} \mathscr{H}_0^{M'}(M),$$

disjoint union over all isomorphism classes of objects $M' \in \mathscr{C}^0_{*\Gamma}$. Note that this is actually a finite union since (for fixed M) $\mathscr{H}_0^{M'}(M)$ is empty for all but finitely many M'. (See 5.14.) Hence

$$\mathscr{H}(M) \textit{ is naturally a projective algebraic variety.}$$

7.3. We have a $\mathbf{C}^*$-action on $\mathscr{H}(M)$ which on each $\mathscr{H}_0^{M'}(M)$ is as in 6.16. Let $\mathscr{H}(M)^{\mathbf{C}^*}$ be the fixed point of this $\mathbf{C}^*$-action. It is clear that $\mathscr{H}(M)^{\mathbf{C}^*}$ is the projective variety consisting of all $S^\dagger$-submodules $\mathscr{W}$ of $S^\dagger \otimes M$ which are also Γ-submodules, such that $\mathscr{W} = \bigoplus_{u \in \mathbf{N}} \mathscr{W}^u$ where $\mathscr{W}^u$ is a subspace of $S^u \otimes M$ for any u, equal to $S^u \otimes M$ for large enough u and such that $(S^\dagger \otimes M)/\mathscr{W} \in \mathscr{C}^0_{*\Gamma}$.

7.4. In many respects, there is a close analogy between the affine Hecke algebra corresponding to, say, $G = PGL_n$ and the modified (degenerate) affine quantized enveloping algebra A corresponding to (I, H) or to Γ.

Similarly, there is a close analogy between the class of projective varieties $\mathscr{B}_u$ (of Borel subgroups of G containing a unipotent element $u \in G$) and the class of projective varieties $\mathscr{H}(M)$ (for $M \in \mathscr{C}^0_{*,\Gamma}$).

Let us illustrate these assertions.

In the affine Hecke algebra one has a canonical basis and corresponding two-sided cells and left cells. The two-sided cells are indexed by the unipotent classes in G and the number of left cells in the two sided cell corresponding to the class of a unipotent element u is the Euler characteristic of $\mathscr{B}_u$.

On the other hand, A has a canonical basis and corresponding two-sided cells and left cells (see [L3]). Conjecturally (see [L3]), the two-sided cells are in bijection with the set $\mathbf{N}[I_*]$ (or equivalently with the set of isomorphism classes of objects $M \in \mathscr{C}^0_{*,\Gamma}$). One can conjecture that the number of left cells contained in the two-sided cell corresponding to M is the Euler characteristic of $\mathscr{H}(M)$. (For type A this is equivalent to the conjecture [L3, 5.7].)

REFERENCES

[BGP] I. N. Bernstein, I. M. Gelfand, and V. A. Ponomarev, Coxeter functors and Gabriel's theorem, *Uspekhi Math. Nauk* **28** (1973), 19–33.

[DR] V. Dlab and C. M. Ringel, "The Preprojective Algebra of a Modulated Graph," Lecture Notes in Math., Vol. 832, pp. 216–231, Springer-Verlag, Berlin/New York, 1980.

[GP] I. M. Gelfand and V. A. Ponomarev, Model algebras and representations of graphs, *Funkt. Anal.* **13** (1979), 1–12.

[LP] L. Le Bruyn and C. Procesi, Semisimple representations of quivers, *Trans. Amer. Math. Soc.* **317** (1990), 585–598.

[L1] G. Lusztig, Quivers, perverse sheaves and enveloping algebras, *J. Amer. Math. Soc.* **4** (1991), 365–421.

[L2] G. Lusztig, Affine quivers and canonical bases, *Publ. Math. I.H.E.S.* **76** (1992), 111–163.

[L3] G. Lusztig, Quantum groups at $v = \infty$, *in* "Functional Analysis on the Eve of the 21st Century: in Honor of I. M. Gelfand, Vol. I," Progr. in Math., Vol. 131, pp. 199–221, Birkhäuser, Basel, 1995.

[M] J. McKay, Graphs, singularities and finite groups, *Proc. Sympos. Pure Math.*, Vol. 37, pp. 183–186, Amer. Math. Soc., Providence, 1980.

[N1] H. Nakajima, Instantons on ALE spaces, quiver varieties and Kac–Moody algebras, *Duke J. Math.* **76** (1994), 365–416.

[N2] H. Nakajima, Quiver varieties and Kac–Moody algebras, *Duke J. Math.* **91** (1998), 515–560.

[N3] H. Nakajima, Lectures on Hilbert schemes of points on surfaces, preprint, 1996.

[W] H. Weyl, "The Classical Groups," Princeton Univ. Press, Princeton, NJ, 1939.

Printed in Belgium